Portfolio Optimization with Tracking-Error Constraints

Philippe Jorion

This article explores the risk and return relationship of active portfolios subject to a constraint on tracking-error volatility (TEV), which can also be interpreted in terms of value at risk. Such a constrained portfolio is the typical setup for active managers who are given the task of beating a benchmark. The problem with this setup is that the portfolio manager pays no attention to total portfolio risk, which results in seriously inefficient portfolios unless some additional constraints are imposed. The development in this article shows that TEV-constrained portfolios are described by an ellipse on the traditional mean–variance plane. This finding yields a number of new insights. Because of the flat shape of this ellipse, adding a constraint on total portfolio volatility can substantially improve the performance of the active portfolio. In general, plan sponsors should concentrate on controlling total portfolio risk.

In typical portfolio delegation, the investor assigns the management of assets to a portfolio manager who is given the task of beating a benchmark. When the investor observes outperformance by the active portfolio, the issue is whether the added value is in line with the risks undertaken. This issue is particularly important when performance fees are involved. Performance fees induce an option-like pattern in the compensation of the manager, who may have an incentive to take on more risk to increase the value of the option.[1] To control this behavior, institutional investors commonly impose a limit on the volatility of the deviation of the active portfolio from the benchmark, which is also known as tracking-error volatility (TEV).

The problem with this setup is that it induces the manager to optimize in only excess-return space while totally ignoring the investor's overall portfolio risk. In an insightful paper, Roll (1992) noted that excess-return optimization leads to the unpalatable result that the active portfolio has systematically higher risk than the benchmark and is not optimal. Jorion (2002) examined a sample of enhanced index funds, which are likely to go through a formal excess-return optimization, and found that such funds have systematically greater risk than the benchmark. Thus, the agency problem is real.

Given these problems, why does the industry maintain this widespread emphasis on controlling tracking-error risk?[2] Roll conjectured that diversifying among managers could mitigate the inherent flaw in TEV optimization, but as I will show later, it does not.

In this article, I investigate whether the agency problem can be corrected with additional restrictions on the active portfolio without eliminating the usual TEV constraint. Thus, because the TEV constraint is so widely used in practice, I take the TEV constraint as given, even though this restriction is not optimal. I derive the constant-TEV frontier in the original mean–variance space.

Traditionally, TEV has been checked after the fact (i.e., from the volatility of historical excess returns), but recently, forward-looking measures of risk, such as value at risk (VAR), have allowed the forecasting of TEV.[3] The essence of VAR is to measure the downside loss for *current* portfolio positions based on the best risk *forecast*. With a distributional assumption for portfolio returns, excess-return VAR is equivalent to a forward-looking measure of TEV. Nowadays, VAR limits are commonly used to ensure that the active portfolio does not stray too far from the benchmark.[4] In addition, pension funds are increasingly allocating their risk through the use of "risk budgets," which can be defined as the conversion of optimal mean–variance allocations to VAR assignments for active managers.[5]

Philippe Jorion is professor of finance at the University of California at Irvine.

The spreading use of VAR systems makes it possible to consider other *ex ante* restrictions on the active portfolio. For this exploration, I analyze the risk and return relationship of active portfolios subject to a TEV constraint.

The primary contribution of this article is the derivation and interpretation of these analytical results. I also illustrate the implications of the analytical results with an example. Apart from Roll's seminal paper, only a few investigations of this important and practical topic are available.[6]

Efficient Frontiers in Absolute and Relative Space

In this section, I review optimization results for the efficient frontiers in absolute and relative spaces.

Setup. Consider a portfolio manager who is given the task of beating an index or benchmark. For this task, the manager must take positions in the assets within the index and, perhaps, other assets. The manager goes about this task as follows.

Define the following variables:

$\mathbf{q}$ = vector of index weights for the sample of N assets

$\mathbf{x}$ = vector of deviations from the index

$\mathbf{q}_P = \mathbf{q} + \mathbf{x}$ = vector of portfolio weights

$\mathbf{E}$ = vector of expected returns

$\mathbf{V}$ = covariance matrix for asset returns

To preserve linearity, assume that net short sales are allowed (i.e., total active weight $\mathbf{q}_i + \mathbf{x}_i$ can be negative for any asset i). Otherwise, the problem generalizes to a quadratic optimization for which there is no closed-form solution.

In practice, the benchmark has positive weight $\mathbf{q}_i$. Generally, it can have negative or zero weights on assets. Thus, the universe of assets can exceed the components of the index. This optimization, however, must include the assets in the benchmark.

Expected returns and variances can now be written in matrix notation as

$\mu_B = \mathbf{q}'\mathbf{E}$ = expected return on the index

$\sigma_B^2 = \mathbf{q}'\mathbf{V}\mathbf{q}$ = variance of index return

$\mu_\in = \mathbf{x}'\mathbf{E}$ = expected excess return

$\sigma_\in^2 = T = \mathbf{x}'\mathbf{V}\mathbf{x}$ = variance of tracking error

Note that these measures are forward-looking measures of risk and return because $\mathbf{x}$ represents current deviations and $\mathbf{V}$ represents the best guess of the covariance matrix over the horizon. Given the initial portfolio value of W_0, the tracking-error VAR is

$$\text{VAR} = W_0 \alpha \sigma_\in, \quad (1)$$

where the parameter α depends on the distributional assumption and the confidence level. Assuming normally distributed returns, for example, means that α is set at 1.645 for a one-tailed confidence level of 95 percent.

The active portfolio expected return and variance are

$$\mu_P = (\mathbf{q} + \mathbf{x})'\mathbf{E} = \mu_B + \mu_\in \quad (2)$$

and

$$\sigma_P^2 = (\mathbf{q} + \mathbf{x})'\mathbf{V}(\mathbf{q} + \mathbf{x}) = \sigma_B^2 + 2\mathbf{q}'\mathbf{V}\mathbf{x} + \mathbf{x}'\mathbf{V}\mathbf{x}. \quad (3)$$

The investment problem is subject to a constraint that the portfolio be fully invested—that is, total portfolio weights $(\mathbf{q} + \mathbf{x})$ must add up to unity. This constraint can be written as

$$(\mathbf{q} + \mathbf{x})'\mathbf{1} = 1, \quad (4)$$

with $\mathbf{1}$ representing a vector of 1's. Because the benchmark weights also add up to unity, the portfolio deviations must add up to zero, which implies that $\mathbf{x}'\mathbf{1}$ is zero. Thus, the active portfolio can be constructed as a position in the index plus a "hedge fund," with positive and negative positions that represent active views.

The Efficient Frontier in Absolute-Return Space. Appendix A reviews the traditional analysis of the mean–variance-efficient frontier, in which there is no risk-free asset. The portfolio allocation problem can be set up as a minimization of σ_P^2 subject to a target expected return of $\mu_P = G$ and full-investment constraint $\mathbf{q}'_P\mathbf{1} = 1$. The solution is given by Equation A2. The efficient set can be described by a hyperbola in the (σ, μ) space, with asymptotes having a slope of $\pm\sqrt{d}$, where d is a function of the efficient-set characteristics. This slope represents the best return-to-risk ratio for this set of assets.

Efficient Frontier in Excess-Return Space. Now, consider the optimization problem in excess-return space. One can trace out the tracking-error frontier by maximizing the expected excess return, $\mu_\in = \mathbf{x}'\mathbf{E}$, subject to a fixed amount of tracking error, $T = \mathbf{x}'\mathbf{V}\mathbf{x}$, and $\mathbf{x}'\mathbf{1} = 0$. The solution, reviewed in Appendix B, is

$$\mathbf{x} = \pm\sqrt{\frac{T}{d}}\mathbf{V}^{-1}(\mathbf{E} - \mu_{MV}\mathbf{1}), \quad (5)$$

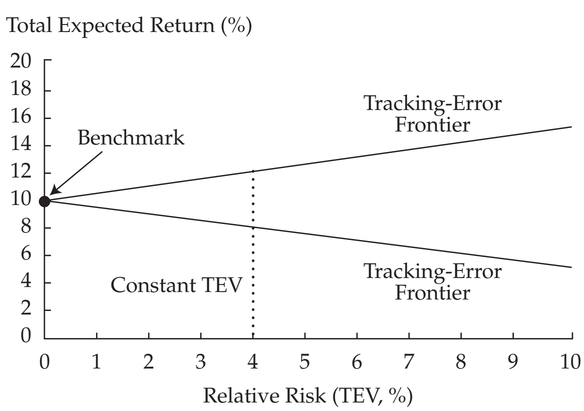
where μ_{MV} is the expected return of the global minimum-variance portfolio. Roll noted that this solution is totally independent of the benchmark because it does not involve **q**. The unexpected result is that active managers pay no attention to the benchmark.[7] In other words, given 5,000 U.S. stocks to choose from, the portfolio manager will take the same active bets whether the index is the S&P 500 or the Russell 2000. This result has major consequences because such behavior is not optimal for the investor.

In mean–volatility space for excess returns, the (upper) efficient frontier is

$$\begin{aligned} \mu_{\in} &= \sqrt{d}\sqrt{T} \\ &= \sqrt{d}\sigma_{\in}, \end{aligned} \tag{6}$$

which is linear in tracking-error volatility, $TEV = \sigma_{\in} = \sqrt{T}$, as shown in **Figure 1**.[8] The benchmark is on the vertical axis because it has zero tracking error.

Figure 1. Tracking-Error Frontier in Excess-Return Space



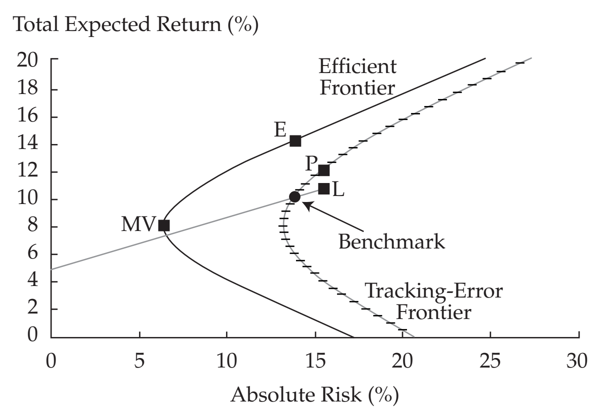
Here, the coefficient $\sqrt{d}$ also represents the information ratio, defined as the ratio of expected excess return to the TEV. The information ratio is commonly used to compare investment managers. Grinold and Kahn (1995), for example, asserted that an information ratio of 0.50 is "good." I chose the efficient-set parameters so that $\sqrt{d}$ would be 0.50.

If the manager is measured solely in terms of excess-return performance, he or she should pick a point on the upper part of this efficient frontier. For instance, the manager may have a utility function that balances expected value added against tracking-error volatility. Note that because the efficient set consists of a straight line, the maximal Sharpe ratio is not a usable criterion for portfolio allocation.

In practice, expected returns are neither observable nor verifiable by the investor. Instead, the portfolio manager is given a constraint on tracking-error volatility, which determines the optimal allocation. This allocation is represented by the intersection of the efficient set with the vertical line representing a constant $\sigma_{\in}$. Figure 1 shows the case of $\sigma_{\in} = 4$ percent. With an information ratio of 0.5, the result is an expected excess return of 200 bps.

TE Frontier in Absolute-Return Space. With this information, one can trace the tracking-error (TE) frontier in traditional absolute-return space as Roll did. **Figure 2** displays this frontier as a line with markings going through the benchmark. Each mark represents a fixed value for TEV (1 percent, 2 percent, and so on). The points represent various portfolios based on data from the global equity indexes provided by Morgan Stanley Capital International. Unhedged total returns were measured in U.S. dollars for the period 1980–2000 for Germany, Japan, the United Kingdom, and the United States. In addition to the equity assets, a fifth asset, the Lehman Brothers U.S. Aggregate Bond Index, was used in the portfolios. The covariance matrix is based on historical data. Expected returns are arbitrary and were chosen so as to satisfy the efficient-set parameters.[9]

Figure 2. Tracking-Error Frontier in Absolute-Return Space



Notes: MV = the global minimum-variance portfolio; E = a portfolio on the efficient frontier with the same level of risk as the benchmark; P = a portfolio with 4 percent tracking error; L = a portfolio leveraged up to have the same risk as Portfolio P.

The graph in Figure 2 shows an unintended effect of TE optimization: Instead of moving toward the true efficient frontier (i.e., up and to the left of the benchmark), the TE frontier moves up and to the right. This outcome increases the total volatility of the portfolio, which is a direct result of focusing myopically on excess returns instead of total returns.

Table 1 displays the characteristics of the efficient frontier and the benchmark for this data set. The expected return and volatility of Benchmark Portfolio B are typical of a well-diversified global equity benchmark. With a 5 percent risk-free rate, its Sharpe ratio is 0.36.

Table 1. Benchmark and Efficient-Set Characteristics

Portfolio	Expected Return (μ)	Volatility (σ)
Benchmark portfolio, B	10.0%	13.8%
Global minimum-variance portfolio, MV	8.0	6.4
Efficient portfolio, E	14.1	13.8
Portfolio with 4% tracking risk, P	12.0	15.4
Leveraged benchmark, L	10.6	15.4

Notes: Portfolio MV achieves the global minimum variance; Portfolio E has the same risk as B but is efficient; Portfolio L leverages up the benchmark to have the same risk as Portfolio P.

The expected return of Portfolio MV is less than that of the benchmark, which should be the case. Otherwise, the index would be grossly inefficient.

Portfolio E is defined as the portfolio on the efficient frontier with the same level of risk as the benchmark (i.e., 14.1 percent). The Portfolio E numbers are typical of the expected performance of active managers because they are based on an information ratio of $\sqrt{d} = 0.50$.

Focus now on Portfolio P with 4 percent tracking risk. Part of the 200 bps increase in expected return of this portfolio relative to the benchmark is illusory because it reflects the higher risk of Portfolio P. To illustrate this point, Figure 2 shows a leveraged portfolio, Portfolio L, achieved with, for instance, stock index futures in such a way that its total risk is also 15.4 percent. Portfolio L is 60 bps above the benchmark—a nonnegligible fraction of the excess performance of 200 bps. So, Figure 2 illustrates the general point that part of the value added of this TEV portfolio is fallacious. The TEV optimization creates an increase in the fund's risk.

Value of Diversification among Managers

Roll conjectured that this increase in risk could be mitigated by diversifying among active managers. Does diversification among managers pay? If the portfolio is equally invested in N managers, the total return on the portfolio, R_P, is given by the return on the benchmark, R_B, plus the average of the active excess returns, $R_{\in,i}$:

$$\begin{aligned} R_P &= \frac{1}{N}\sum_{i=1}^{N}(R_B + R_{\in,i}) \\ &= R_B + \frac{1}{N}\sum_{i=1}^{N} R_{\in,i}. \end{aligned} \tag{7}$$

The total portfolio variance can be derived from Equation B6 in Appendix B. If all active excess positions are assumed to have the same tracking risk and information ratio, the result is

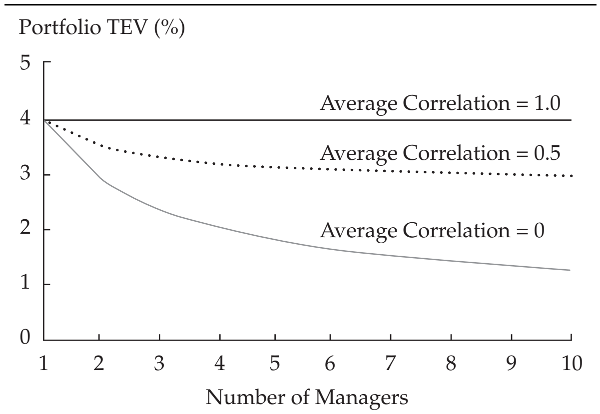
$$\begin{aligned} \sigma_P^2 &= \sigma_B^2 + \frac{2}{N}\sum_{i=1}^{N}\operatorname{cov}(R_B, R_{\in,i}) + \frac{1}{N^2}\sigma^2\left(\sum_{i=1}^{N} R_{\in,i}\right) \\ &= \sigma_B^2 + 2\sqrt{\frac{T}{d}}(\mu_B - \mu_{MV}) + \sigma^2\left(\frac{1}{N}\sum_{i=1}^{N} R_{\in,i}\right). \end{aligned} \tag{8}$$

The second term in Equation 8 represents the covariance between the index and the average portfolio deviation. The covariance is positive and does not depend on the number of managers. The third term, in contrast, is affected by diversification. It represents the variance of the portfolio tracking error. If all excess returns are assumed to have the same correlation, ρ, with each other, this term can be written as

$$\sigma\left(\frac{1}{N}\sum_{i=1}^{N} R_{\in,i}\right) = \sigma_{\in}\sqrt{\frac{1}{N} + \left(1 - \frac{1}{N}\right)\rho}. \tag{9}$$

The variance term decreases with more managers or lower correlations. **Figure 3** shows, however, that with realistic data, the rate of decrease is

Figure 3. Decrease in Tracking Risk with Multiple Managers



small. With 10 managers and $\rho = 0.5$, for instance, the volatility of total tracking error decreases only from 4.0 percent to 3.0 percent. Thus, diversifying among managers is not likely to mitigate the inherent flaw in tracking-error optimization.

Constant-TEV Frontier

Now that I have shown the drawbacks of TEV optimization, the issue is whether additional constraints can be used to improve the performance of TEV-constrained portfolios. The first step is to characterize the locus of points that correspond to a TEV constraint in the original MV space. The optimization can be written as

$$\begin{aligned} &\text{Maximize } \mathbf{x}'\mathbf{E} \\ &\text{subject to} \quad \mathbf{x}'\mathbf{1} = 0 \\ &\qquad \mathbf{x}'\mathbf{V}\mathbf{x} = T \\ &\qquad (\mathbf{q}+\mathbf{x})'\mathbf{V}(\mathbf{q}+\mathbf{x}) = \sigma_P^2. \end{aligned}$$

The first constraint sets the sum of portfolio deviations to zero. The second constraint sets the tracking-error variance to a fixed amount T. Finally, the third constraint forces the total portfolio variance to be equal to a fixed value σ_P^2. This number can be varied to trace out the constant-TEV frontier. The solution is given in Appendix C.

For what follows, I define the quantities $\Delta_1 = \mu_B - \mu_{MV} \geq 0$ and $\Delta_2 = \sigma_B^2 - \sigma_{MV}^2 \geq 0$, which characterize the expected return and variance of the index in excess of that of the minimum-variance portfolio. These quantities play a central role in the description of the TEV frontier. For this data set, $\Delta_1 = 2$ percent and $\Delta_2 = 0.0149$.

The relationship between expected return and variance for a fixed TEV turns out to be an ellipse—Equation C6 in Appendix C. The ellipse is somewhat distorted in (σ,μ) space and is illustrated in **Figure 4**.

Next, **Figure 5** shows the effect of changing TEV on this frontier. When $\sigma_\in$ is zero, the ellipse collapses to a single point, the benchmark. As $\sigma_\in$ increases, the size of the ellipse increases. The left side of the ellipse moves to the left and becomes tangent to the efficient-set parabola at one point.

The first tangency occurs at $\sigma_\in = \sqrt{\Delta_2 - \Delta_1^2/d} = 11.5\%$. After that point, two tangency points occur. As $\sigma_\in$ increases, the ellipse moves to the right. For $\sigma_\in = 2\sqrt{\Delta_2 - \Delta_1^2/d} = 23.0\%$, the ellipse passes through the index itself. All active portfolios with TEV constraints and positive excess returns must have greater risk than the index.

These analytical results, proved in Appendix C, show that tracking-error volatility should be chosen carefully. If TEV values are set too high, maintaining a level of risk similar to that of the benchmark is impossible.

Figure 4. Frontier with Constant Tracking-Error Volatility

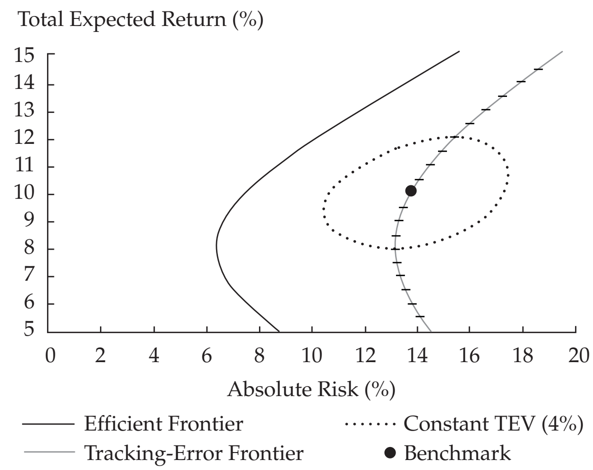
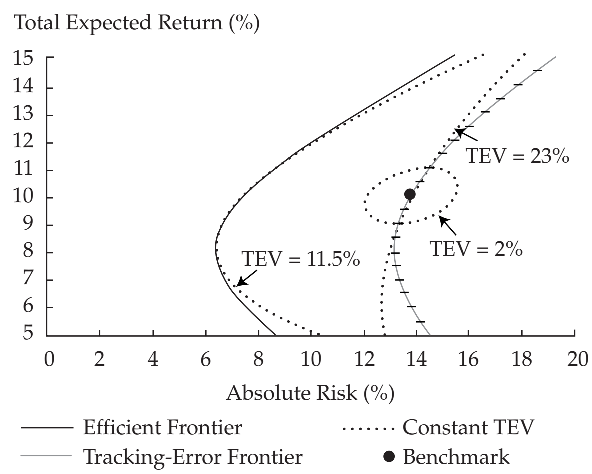


Figure 5. Constant-TEV Frontiers for Various Steps



Following these results, the next question is whether the investor might be able to induce the active manager to move closer to the efficient frontier by imposing additional constraints.

Moving Closer to the Efficient Frontier

Could imposing additional restrictions on the active manager bring the portfolio closer to the efficient frontier?

Risk–Return Trade-Off. One solution would be for the investor to provide a manager with the investor's risk–return trade-off. The manager would then optimize the investor's utility subject to the TEV constraint. For instance, the problem can be set up as follows:

$$\text{Maximize } U(\mu_P, \sigma_P) = \mu_P - \frac{1}{(2t)}\sigma_P^2, \tag{10}$$

where t is the investor's risk tolerance subject to the TEV constraint.

The problem with this approach is that it is impractical to verify. *Ex ante*, the manager may not be willing to disclose expected returns. *Ex post*, realized returns are enormously noisy measures of expected returns. Instead, it is much easier to constrain the risk profile, either before or after the fact—which is no doubt why investors give managers tracking-error constraints.

Armed with the equation for a constant-TEV frontier (Equation C6), we can now explore the effectiveness of imposing additional restrictions. One such constraint, explored by Roll, is to impose a beta of 1. But we can do even more.

Constraint on Total Risk. The investor could specify that the portfolio risk be equal to that of the index itself:

$$\sigma_P^2 = \sigma_B^2. \tag{11}$$

From Equation 3, this constraint implies that $2\mathbf{q}'\mathbf{V}\mathbf{x} = -T$, or that the benchmark deviations must have a negative covariance with the index. Figure 5 shows that when TEV is about 12 percent, such a constraint on absolute volatility can bring the portfolio much closer to the efficient set. Imposing an additional restriction on the manager, however, must decrease expected returns. The cost can be derived from Equation C16 in Appendix C. The issue is whether this restriction is really harmful.

The shape of the constant-TEV frontier in Figure 4 suggests that the loss from this restriction may not be large. The top part of the ellipse is rather flat. The effects of a constraint on total volatility are illustrated in **Table 2**, which reports the drop in expected return and the associated reduction in volatility for various levels of σ_{MV} and of Δ_1. The ratio of the drop in μ to that in σ can be viewed as the cost of the constraint.

Table 2 shows that when $\Delta_1 = 0$ percent (that is, $\mu_B = \mu_{MV}$), $\sigma_{MV} = 8$ percent, and the TEV is set at 4 percent, imposing a constraint on total volatility leads to a loss of expected return of only 0.03 percentage point (pp). The risk reduction gained in exchange is 0.57 pps, so the ratio is 0.06. When $\Delta_1 =$ 2 percent and other settings are the same as previously, the loss of expected return is 0.29 pps in exchange for a risk reduction of 1.65 pps, for a ratio of 0.18.

These return-to-risk ratios compare favorably with an intrinsic information ratio (return-to-risk) of 0.50. Thus, the cost of the additional constraint on total volatility is low.

The conditions under which this constraint is most useful can also be identified from Table 2. The conditions depend on the size of the tracking-error constraint and the efficiency of the benchmark. First, the lower the TEV, the more helpful the constraint. Indeed, the ratio of the drop in expected return to drop in volatility decreases as one moves from the right of the table to the left. Second, the less efficient the index, the better the constraint. The cost of the constraint decreases when $\Delta_1 = \mu_B - \mu_{MV}$ is low, which means that the expected return on the index is low. The cost also decreases when σ_{MV} is low relative to σ_B, which means that the risk of the index is large relative to the efficient frontier.

Hence, imposing a constraint on the total risk appears sensible precisely in situations where the benchmark is relatively inefficient. If the active manager is confident that he or she can add value, the manager should easily accept an additional constraint on total portfolio risk.

Illustration of Portfolio Positions. The results obtained so far depend only on the efficient-set parameters and the characteristics of the benchmark. They hold for any number of assets. **Table 3** shows how these numbers could be achieved with hypothetical expected returns for the four global equity markets and the Lehman Brothers U.S. bond index. Table 3 reports expected returns and positions for three portfolios—the benchmark, the 4 percent TEV-constrained active portfolio, and the portfolio with an additional constraint that the total risk must equal that of the benchmark.

The information ratio of 0.5 was driven primarily by the dispersion in expected returns, as shown in that column. I chose high expected returns for German and U.K. equities, moderate returns for U.S. equities, and low expected returns for Japanese equities. I set the expected return from U.S. bonds at 8 percent. The next column shows the positions for the benchmark; these weights correspond to those in the global stock index in 2000. As before, the index is expected to return 10 percent.

The next two columns display positions in the usual TEV-constrained portfolio. To increase returns, the active manager increases the position in German and U.K. equities and decreases the position in Japanese equities, U.S. equities, and bonds. This move increases the expected return by 200 bps. But, unfortunately, the total risk also increases—from 13.8 percent to 15.4 percent.

The last two columns report positions for the TEV-constrained portfolio with an additional constraint on total risk. This portfolio does indeed have

Table 2. Effect of Additional Constraint on Return and Risk by TEV

Δ_1 and σ_{MV}	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%
Δ_1 = 0%										
	Drop in μ (percentage points, pps)									
σ_{MV} = 6%	0.00	0.00	–0.01	–0.03	–0.05	–0.09	–0.14	–0.21	–0.31	–0.43
σ_{MV} = 8%	0.00	0.00	–0.01	–0.03	–0.06	–0.11	–0.18	–0.26	–0.38	–0.53
σ_{MV} = 10%	0.00	–0.01	–0.02	–0.05	–0.09	–0.16	–0.25	–0.38	–0.54	–0.76
	Drop in σ (pps)									
	–0.04	–0.14	–0.32	–0.57	–0.88	–1.25	–1.68	–2.16	–2.68	–3.25
	Ratio: drop in μ/σ									
σ_{MV} = 6%	0.01	0.02	0.03	0.05	0.06	0.07	0.09	0.10	0.11	0.13
σ_{MV} = 8%	0.01	0.03	0.04	0.06	0.07	0.09	0.10	0.12	0.14	0.16
σ_{MV} = 10%	0.02	0.04	0.06	0.08	0.10	0.12	0.15	0.17	0.20	0.23
Δ_1 = 1%										
	Drop in μ (pps)									
σ_{MV} = 6%	–0.01	–0.03	–0.06	–0.10	–0.17	–0.25	–0.35	–0.47	–0.63	–0.81
σ_{MV} = 8%	–0.01	–0.04	–0.07	–0.13	–0.20	–0.30	–0.43	–0.58	–0.77	–1.00
σ_{MV} = 10%	–0.02	–0.05	–0.10	–0.18	–0.28	–0.42	–0.60	–0.82	–1.09	–1.42
	Drop in σ (pps)									
	–0.18	–0.43	–0.74	–1.12	–1.55	–2.03	–2.56	–3.13	–3.74	–4.39
	Ratio: drop in μ/σ									
σ_{MV} = 6%	0.06	0.07	0.08	0.09	0.11	0.12	0.14	0.15	0.17	0.19
σ_{MV} = 8%	0.07	0.08	0.10	0.11	0.13	0.15	0.17	0.19	0.21	0.23
σ_{MV} = 10%	0.10	0.12	0.14	0.16	0.18	0.21	0.23	0.26	0.29	0.32
Δ_1 = 2%										
	Drop in μ (pps)									
σ_{MV} = 6%	–0.03	–0.08	–0.15	–0.24	–0.35	–0.48	–0.64	–0.84	–1.06	–1.32
σ_{MV} = 8%	–0.04	–0.10	–0.18	–0.29	–0.42	–0.59	–0.79	–1.02	–1.30	–1.62
σ_{MV} = 10%	–0.06	–0.14	–0.26	–0.41	–0.60	–0.83	–1.11	–1.44	–1.83	–2.28
	Drop in σ (pps)									
	–0.32	–0.71	–1.15	–1.65	–2.19	–2.77	–3.40	–4.06	–4.74	–5.46
	Ratio: drop in μ/σ									
σ_{MV} = 6%	0.10	0.12	0.13	0.14	0.16	0.17	0.19	0.21	0.22	0.24
σ_{MV} = 8%	0.13	0.14	0.16	0.18	0.19	0.21	0.23	0.25	0.27	0.30
σ_{MV} = 10%	0.18	0.20	0.23	0.25	0.27	0.30	0.33	0.35	0.38	0.42

lower volatility than the TEV-constrained portfolio; in fact, its total risk is 13.8 percent, equal to that of the benchmark. The most interesting aspect of the table, however, is that achieving this reduction in risk comes at a very low cost: The expected return is only marginally lower than it was before adding risk control (i.e., 11.8 percent instead of 12 percent). The strategy to achieve this outcome was to short more U.S. equities and move the proceeds into U.S. bonds with their low total risk.

Thus, adding a constraint on total risk preserves most of the benefits of active management while it remedies the inherent flaw in excess-return optimization.

Conclusions

The common practice in the investment management industry is to control the risk of active managers by imposing a constraint on tracking error.

Table 3. Illustrative Positions

		Positions				
	Expected Return	Benchmark Weight	TEV-Constrained Portfolio		TEV-Constrained and Risk-Constrained Portfolio	
Asset	(μ)	(**q**)	**x**	**q** + **x**	**x**	**q** + **x**
German equities	14.7%	6.6%	10.5%	17.1%	9.8%	16.4%
Japanese equities	5.7	17.5	–10.6	6.9	–13.4	4.1
U.K. equities	14.7	12.2	17.5	29.7	16.3	28.5
U.S. equities	9.8	63.7	–6.7	57.0	–16.8	46.9
U.S. bonds	8.0	0.0	–10.7	–10.7	4.1	4.1
Total weight		100.0%	0.0%	100.0%	0.0%	100.0%
Portfolio		Total	Excess	Total	Excess	Total
Expected return		10.0%	2.0 pps	12.0%	1.8 pps	11.8%
Risk		13.8	4.0	15.4	4.0	13.8

Note: The variable **x** represents the vector of deviations from the index, and the term **q** + **x** is the vector of portfolio weights.

This setup, however, is seriously inefficient. When myopically focusing on excess returns, the active manager ignores the total risk of the portfolio. As a result, optimization of excess returns that includes the benchmark assets will always increase total portfolio risk relative to the benchmark.

This outcome is reinforced by the widespread use of information ratios as performance measures. Because information ratios consider only tracking-error risk, a focus on information ratios ignores total portfolio risk.

This issue has major consequences for performance measurement: Part of the value added by active managers acting in this fashion is illusory; it could be naively obtained by leveraging up the benchmark.

Because the industry continues to emphasize tracking-error constraints and information ratios, I considered in this article what can be done to mitigate the inefficiency of using TEV constraints. I derived analytical solutions for the risk–return relationship of portfolios subject to a TEV constraint. And I showed that the constraint is described by an ellipse in the usual mean–variance space. This finding allowed exploration of the effect of imposing additional constraints on the active manager.

The simplest constraint is to force total portfolio volatility to be no greater than that of the benchmark. With the advent of forward-looking risk measures, such as VAR, such a constraint is easy to set up. I showed that because of the flat shape of the ellipse, adding such a constraint can substantially improve the performance of the active portfolio. The risk-control constraint is most beneficial in situations with low values for the admissible TEV or when the benchmark is relatively inefficient.

In summary, my first prescription is to discard TEV optimization and focus instead on total risk. Some indications are that pension plans with advanced risk management systems are indeed moving in this direction.[10] If TEV constraints must be kept in place, my recommendation is to impose an additional constraint on total volatility. This article provides the tools to do so.

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Appendix A. Mean–Variance-Efficient Frontier

In the derivation of the conventional efficient frontier without a risk-free asset, G is the target expected return. The allocation problem involves a constrained minimization of the portfolio variance over the weights q_P:

$$\text{Minimize } \mathbf{q}'_P \mathbf{V} \mathbf{q}_P$$

$$\text{subject to} \quad \mathbf{q}'_P \mathbf{1} = 1$$

$$\mathbf{q}'_P \mathbf{E} = G.$$

Following Merton (1972), define the efficient-set constants as

$$a = \mathbf{E}'\mathbf{V}^{-1}\mathbf{E};$$

$$b = \mathbf{E}'\mathbf{V}^{-1}\mathbf{1};$$

$$c = \mathbf{1}'\mathbf{V}^{-1}\mathbf{E};$$

$$d = a - \frac{b^2}{c}.$$

The efficient frontier can be fully defined by two portfolios—one that minimizes the variance (the MV portfolio) and another (the TG portfolio) that is tangent to the efficient set and that maximizes the return-to-risk ratio—with the weights

$$\mathbf{q}_{MV} = \mathbf{V}^{-1}\frac{\mathbf{1}}{c}; \tag{A1}$$

$$\mathbf{q}_{TG} = \mathbf{V}^{-1}\frac{\mathbf{E}}{b}. \tag{A2}$$

The expected return, E, and variance, V, of the two portfolios are

$$E_{TG} = \frac{a}{b},$$

$$V_{TG} = \frac{a}{b^2}$$

and

$$E_{MV} = \frac{b}{c},$$

$$V_{MV} = \frac{1}{c}.$$

When the covariance matrix is positive definite, the constants a and c must be positive. In addition, the efficient set is meaningful when the expected return on the tangent portfolio is greater than the return on the minimum-variance portfolio, which implies that $d > 0$.

Taking the Lagrangian and setting the partial derivatives of it to zero, one finds that the allocations for any portfolio can be described as a linear combination of the two portfolios:

$$\mathbf{q}_P = \left(\frac{a - bG}{d}\right)\mathbf{q}_{MV} + \left(\frac{Gb - b^2/c}{d}\right)\mathbf{q}_{TG}. \tag{A3}$$

Computing the variance and setting G equal to μ_P, one finds that the efficient set is represented by

$$\begin{aligned}\sigma_P^2 &= \frac{a}{dc} - \frac{2b}{dc}\mu_P + \frac{1}{d}\mu_P^2 \\ &= \frac{1}{d}\left(\mu_P - \frac{b}{c}\right)^2 + \frac{1}{c} \\ &= \frac{1}{d}(\mu_P - \mu_{MV})^2 + \sigma_{MV}^2,\end{aligned} \tag{A4}$$

which represents a parabola in the (σ^2, μ) space or a hyperbola in the (σ, μ) space with asymptotes having a slope of $\pm\sqrt{d}$. This slope represents the best return-to-risk ratio for this set of assets.

Appendix B. Tracking-Error Frontier

This discussion presents the derivation of the shape of the tracking-error frontier in the *excess* mean–variance space (i.e., relative to a benchmark).

One must assume that the benchmark is not on the efficient set; otherwise, there would be no rationale for active management. In addition, the expected return on the benchmark is assumed to be greater than or equal to that of the minimum-variance portfolio: $\mu_B \geq \mu_{MV} = b/c$. If this condition were not satisfied, the benchmark would be grossly inefficient because the investor could pick another index with the same risk but higher expected return.

Consider a maximization of portfolio excess return over the deviations $\mathbf{x}$ from the benchmark:

$$\begin{aligned}&\text{Maximize } \mathbf{x}'\mathbf{E} \\ &\text{subject to} \quad \mathbf{x}'\mathbf{1} = 0 \\ &\qquad\qquad\quad \mathbf{x}'\mathbf{V}\mathbf{x} = \sigma_\in^2 = T.\end{aligned}$$

Set up the Lagrangian L using the multipliers λ

$$L = \mathbf{x}'\mathbf{E} + \lambda_1(\mathbf{x}'\mathbf{1} - 0) + 0.5\lambda_2(\mathbf{x}'\mathbf{V}\mathbf{x} - T). \tag{B1}$$

Taking partial derivatives with respect to $\mathbf{x}$ and setting L to zero provides the solution of the form

$$\mathbf{x} = \frac{-1}{\lambda_2}\mathbf{V}^{-1}(\mathbf{E} + \lambda_1\mathbf{1}). \tag{B2}$$

Selecting the values of the λ's so that the two constraints are satisfied produces

$$\mathbf{x} = \pm\sqrt{\frac{T}{d}}\,\mathbf{V}^{-1}\left(\mathbf{E} - \frac{b}{c}\mathbf{1}\right). \tag{B3}$$

Note that deviations $\mathbf{x}$ do not depend on the benchmark. This unexpected result arises from the fact that the portfolio manager considers only tracking-error risk.

Solving now for the portfolio expected excess return produces

$$\mu_\in = \pm\sqrt{d}\,\sigma_\in, \tag{B4}$$

where the upper part is a straight line in tracking-error space.

This equation can be translated back into the usual mean–variance space as follows:

$$\begin{aligned}\mu_P &= (\mathbf{q} + \mathbf{x})'\mathbf{E} \\ &= \mu_B \pm \sqrt{d}\sqrt{T};\end{aligned} \tag{B5}$$

$$\begin{aligned}\sigma_P^2 &= (\mathbf{q} + \mathbf{x})'\mathbf{V}(\mathbf{q} + \mathbf{x}) \\ &= \sigma_B^2 \pm 2\sqrt{\frac{T}{d}}(\mu_B - \mu_{MV}) + T.\end{aligned} \tag{B6}$$

After substitution for T, Equations B5 and B6 represent a hyperbola in the (σ_P, μ_P) space with the same asymptotes as the conventional efficient frontier. When the benchmark is efficient, this hyperbola collapses to the efficient frontier.

Appendix C. TEV Frontier in Absolute-Return Space

In this appendix, the shape of the constant-TEV frontier in the original mean–variance space is derived. Define the quantities

$$\Delta_1 = \mu_B - b/c = \mu_B - \mu_{MV}$$

and

$$\Delta_2 = \sigma_B^2 - 1/c = \sigma_B^2 - \sigma_{MV}^2,$$

which characterize, respectively, the expected return and variance of the index in excess of the return and variance of the minimum-variance portfolio. The term Δ_2 is always positive because MV is by definition the variance of the minimum-variance portfolio. The term Δ_1 should also be positive, as explained in Appendix B.

Derivation of the Frontier. The first theorem (concerning the shape of the TEV-constrained frontier) is as follows:

Theorem 1: The constant-TEV frontier is an ellipse in the (σ^2, μ) space centered at μ_B and $\sigma_B^2 + T$. With the deviations from the center defined as $y = \sigma_P^2 - \sigma_B^2 - T$ and $z = \mu_P - \mu_B$, the constant-TEV frontier is given by Equation C6.

Consider a maximization, or equivalently a minimization, over $\mathbf{x}$:

Maximize $\mathbf{x'E}$

subject to

$$\mathbf{x'1} = 0$$
$$\mathbf{x'Vx} = T$$
$$(\mathbf{q} + \mathbf{x})'\mathbf{V}(\mathbf{q} + \mathbf{x}) = \sigma_P^2.$$

Set up the Lagrangian as

$$L = \mathbf{x'E} + \lambda_1(\mathbf{x'1} - 0) + 0.5\lambda_2(\mathbf{x'Vx} - T) + 0.5\lambda_3(\mathbf{x'Vx} + 2\mathbf{q'Vx} + \mathbf{q'Vq} - \sigma_P^2). \quad \text{(C1)}$$

Taking partial derivatives with respect to $\mathbf{x}$ and setting L to zero provides the solution of the form:

$$\mathbf{x} = \frac{-1}{\lambda_2 + \lambda_3}\mathbf{V}^{-1}(\mathbf{E} + \lambda_1\mathbf{1} + \lambda_3\mathbf{Vq}). \quad \text{(C2)}$$

Now, select the values of the λ's so that the three constraints are satisfied. The result is

$$b + \lambda_1 c + \lambda_3 = 0;$$

$$a + \lambda_1^2 c + \lambda_3^2\sigma_B^2 + 2b\lambda_1 + 2\mu_B\lambda_3 + 2\lambda_1\lambda_3 = T(\lambda_2 + \lambda_3)^2;$$

$$\mu_B + \lambda_1 + \lambda_3\sigma_B^2 = \frac{(\sigma_P^2 - \sigma_B^2 - T)(\lambda_2 + \lambda_3)}{2}.$$

Define y as $\sigma_P^2 - \sigma_B^2 - T$. Solving for the λ's produces

$$\lambda_3 = -\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2}\sqrt{\frac{(d\Delta_2 - \Delta_1^2)}{(4T\Delta_2 - y^2)}}; \quad \text{(C3)}$$

$$\lambda_1 = -\frac{(\lambda_3 + b)}{c}; \quad \text{(C4)}$$

$$\lambda_2 + \lambda_3 = \pm(-2)\sqrt{\frac{(d\Delta_2 - \Delta_1^2)}{(4T\Delta_2 - y^2)}}. \quad \text{(C5)}$$

Now, define $z = \mu_P - \mu_B$. Replacing terms in Equation C2, compute $\mathbf{x'E}$. The relationship between y and z can be derived as

$$dy^2 + 4\Delta_2 z^2 - 4\Delta_1 yz - 4T(d\Delta_2 - \Delta_1^2) = 0. \quad \text{(C6)}$$

For a quadratic equation of the type $Ay^2 + Bz^2 + Cyz + F = 0$, Equation C6 represents an ellipse when the term

$$AB - \left(\frac{1}{4}\right)C^2 = d(4\Delta_2) - \left(\frac{1}{4}\right)(-4\Delta_1)^2 = 4(d\Delta_2 - \Delta_1^2)$$

is strictly positive. This term must be positive when the benchmark is within the efficient set. The efficient set represented by Equation A3 requires that $d\Delta_2 - \Delta_1^2 \geq 0$.

When $\Delta_1 = \mu_B - \mu_{MV} > 0$, the main axis of the ellipse is not horizontal but, instead, has a positive slope. If the expected return on the benchmark happens to be equal to that of the minimum-variance portfolio, the ellipse is horizontal.

Properties of the Frontier. The properties of the ellipse that describes portfolios with constant tracking-error volatility in the mean–variance space can be further analyzed.

- *Centering of ellipse*. The vertical center of the ellipse is the expected return of the index, μ_B. The horizontal center of the ellipse is displaced to the right by the amount of TEV, $\sigma_B^2 + T$. Thus, increasing tracking error shifts the center of the ellipse to regions of higher total risk.
- *Maximum and minimum expected returns*. Because the maximum and minimum expected excess returns are obtained from the TEV frontier in excess-return space, the absolute maximum and minimum expected returns on the constant-TEV line are achieved at the intersection with the tracking-error frontier. From Equation B5, this intersection is

$$\mu_P = \mu_B \pm \sqrt{dT}. \quad \text{(C7)}$$

Faced with only a TEV constraint, the active manager will simply maximize the expected return for a given T. The problem is that this practice can substantially increase total portfolio risk. At this point, the variance is

$$\sigma_P^2 = \sigma_B^2 + T + 2\Delta_1\sqrt{\frac{T}{d}}. \quad \text{(C8)}$$

Hence, active portfolio risk increases not only directly with TEV but also with the quantity $\Delta_1 = \mu_B - \mu_{MV}$. Increasing Δ_1 means, with a fixed σ_B^2, that the benchmark becomes more efficient. If so, active management must substantially increase portfolio risk.

■ *Maximum and minimum variance*. With constant TEV, the absolute maximum and minimum values for the variance along the ellipse are given by

$$\sigma_P^2 = \sigma_B^2 + T \pm 2\sqrt{T(\sigma_B^2 - \sigma_{MV}^2)}, \quad \text{(C9)}$$

which does not depend on expected returns. Hence, the width of the ellipse depends not only on TEV but also on the distance between the variance of the index and that of the global minimum-variance portfolio.

Effect of Changing TEV. Consider now the effect of changing TEV on these limits. The first part of the second theorem (concerning the minimum TEV for contact with the efficient set) is as follows:

> *Theorem 2a: The constant-TEV frontier achieves first contact with the efficient set when $\sigma_\in$ is equal to $\sqrt{\Delta_2 - (\Delta_1^2/d)}$; this point occurs for a level of expected return equal to that of the benchmark.*

Figure 5 shows that portions of the ellipse touch the efficient set for large values of TEV. The contact points between the ellipse and the efficient-set parabola can be defined. Using σ_P^2 and y from Equation A4 in Equation C6 gives

$$\begin{aligned} 0 = {} & d\left[\frac{1}{d}(z+\Delta_1)^2 + \frac{1}{c} - \sigma_B^2 - T\right]^2 + 2\Delta_2 z^2 \\ & - 4\Delta_1 z\left[\frac{1}{d}(z+\Delta_1)^2 + \frac{1}{c} - \sigma_B^2 - T\right] \\ & - 4T(d\Delta_2 - \Delta_1^2), \end{aligned} \quad \text{(C10)}$$

which is a quartic equation in z.[11] After simplification, and defining $k = dT - d\Delta_2 + \Delta_1^2$, Equation C10 gives

$$\begin{aligned} & z^4 - 2z^2(dT - d\Delta_2 + \Delta_1^2) \\ & \quad + (dT - d\Delta_2 + \Delta_1^2)^2 = (z^2 - k)^2 = 0. \end{aligned} \quad \text{(C11)}$$

Equation C11 has a solution when $k = dT - d\Delta_2 + \Delta_1^2 \geq 0$ or when T is large enough. When no solution exists, the curves do not intersect; only one contact point occurs—when $k = 0$ or when the tracking-error variance is

$$\sigma_\in^2 = T_A = \Delta_2 - \frac{\Delta_1^2}{d}, \quad \text{(C12)}$$

at which point contact occurs for $z = 0$ or when $\mu_P = \mu_B$. In other words, first contact with the efficient set occurs on the horizontal from the index. For the example in Figure 5, this point arrives at TEV = 11.5 percent. As T increases, two contact points result, for which $z = \pm\sqrt{k}$.

The second part of Theorem 2 (concerning TEV and minimum risk) is as follows:

> *Theorem 2b: When $\sigma_\in = \sqrt{\Delta_2}$, the constant-TEV frontier achieves a minimum level of risk equal to that of the global minimum-variance portfolio.*

Equation C9 can also be written as

$$\sigma_P^2 - \sigma_{MV}^2 = (\sqrt{T} \pm \sqrt{\Delta_2})^2. \quad \text{(C13)}$$

The portfolio achieves minimum risk when $\sigma_P^2 = \sigma_{MV}^2$ or when

$$\sigma_\in^2 = T_B = \Delta_2. \quad \text{(C14)}$$

At this point, the lowest portfolio variance along the ellipse coincides with the global minimum-variance portfolio. In the example, this point is reached at TEV = 12.2 percent.

The third part of Theorem 2 (which concerns the index outside the TEV frontier) is as follows:

> *Theorem 2c: When $\sigma_\in = 2\sqrt{\Delta_2 - (\Delta_1^2/d)}$, the constant-TEV frontier passes through the benchmark itself. Above this value, the benchmark is no longer within the constant-TEV frontier.*

The ellipse passes through the benchmark position when (with $y = -T$ and $z = 0$ in Equation C6) $dT^2 - 4T(d\Delta_2 - \Delta_1^2) = 0$, which implies that

$$T_C = 4\left(\Delta_2 - \frac{\Delta_1^2}{d}\right). \quad \text{(C15)}$$

In Figure 5, this point is reached for TEV = 23.0 percent. Beyond this point, all TEV-constrained portfolios with positive excess returns must have greater risk than the index.

Finally, the fourth part of Theorem 2 (which concerns benchmark risk and the TEV frontier) is as follows:

> *Theorem 2d: When $\sigma_\in = 2\sqrt{\Delta_2}$, the constant-TEV frontier achieves a minimum level of risk equal to that of the benchmark. Above this value, any constant-TEV portfolio has risk greater than that of the benchmark.*

Increasing T further moves the ellipse back to the right. In particular, when

$$\sigma_{\in} = T_D = 4\Delta_2, \tag{C16}$$

then σ_P^2 is equal to σ_B^2. In the example, this point is achieved for TEV = 24.4 percent. Beyond this point, all TEV-constrained portfolios must have greater risk than the index.

Constraint on Total Risk. The equation of the TEV ellipse can be used to compute the expected return when total risk is set to the risk of the index. Evaluating Equation C6 at $y = \sigma_P^2 - \sigma_B^2 - T = -T$ results in

$$\mu_P - \mu_B = -T\frac{\Delta_1}{2\Delta_2} \pm \sqrt{T\left(d - \frac{\Delta_1^2}{\Delta_2}\right)\left(1 - \frac{T}{4\Delta_2}\right)}. \tag{C17}$$

Notes

1. This issue arises in the case of hedge funds, for instance, which typically charge a variable fee of 20 percent of profits. For a good introduction to the major issues surrounding performance fees, see Davanzo and Nesbitt (1987), Grinold and Rudd (1987), and Kritzman (1987).
2. Going further than this question, Admati and Pfleiderer (1997) examined the rationale for benchmark-adjusted compensation schemes. They argued that such schemes are generally inconsistent with optimal risk sharing and do not help in solving potential contracting problems with the portfolio manager.
3. See Jorion (2000) for a detailed analysis of VAR.
4. Another issue in portfolio control is who should be given authority to control tracking risk. Possible candidates are the investment manager, the plan sponsor, the custodian, or outside consultants. One could argue that risk should be

controlled by the investment manager. After all, the manager should already have in place a risk measurement system that gives the tracking error of the active portfolio. The manager should also have the best understanding of the instruments in the portfolio. Thomas (2000) argued, however, that this delegation of risk control to the manager creates a conflict of interest for the manager and that risk control is best performed by a disinterested party.

5. For an introduction to risk budgeting, see Chow and Kritzman (2001). Lucas and Klaassen (1998) also discuss the link between portfolio optimization and VAR.
6. The closest paper is that of Leibowitz, Kogelman, and Bader (1992), who discussed the application of the shortfall approach to portfolio choice for a pension fund. In their case, the tracking-error volatility was replaced by "surplus return," which was defined relative to the liabilities. Their paper entailed another constraint, however—a linear relationship between expected returns and volatility—and involved a simple setup with only two risky assets. In addition, Leibowitz et al. presented no closed-form solutions. Chow (1995) argued that the objective function should account for total risk but also tracking-error risk. Rudolf, Wolter, and Zimmermann (1999) compared various linear models to minimize tracking error. Ammann and Zimmermann (2001) examined the relationship between limits on TEV and deviations from benchmark weights.
7. In practice, the active positions will depend on the benchmark if the mandate has short-selling restrictions on total weights. Assets with low expected returns can be shorted only up to the extent of the (long) position in the benchmark.
8. With restrictions that the total portfolio weights cannot be negative, $\mathbf{q}_i + \mathbf{x}_i \geq 0$, the efficient frontier starts as a straight line, then becomes concave as some of the restrictions become binding, $\mathbf{x}_i = -\mathbf{q}_i$. It then flattens out until the whole active portfolio is invested in the asset with the highest expected return.
9. In practice, substantial estimation error in expected returns can result when estimates are based on historical data. Therefore, I did not use historical data but, instead, adjusted expected returns to achieve a "reasonable" information ratio. As Michaud (1989) showed, the optimal portfolio is quite sensitive to errors in expected returns. Jorion (1992) showed that when data are taken from historical observations, the variability in the weights can be gauged from simulations based on the original sample. In contrast, the covariance matrix can be more precisely estimated. Chan, Karceski, and Lakonishok (1999) showed that for optimization purposes, the covariance matrix contains substantial predictability.
10. For instance, ABP, the Dutch pension plan that has $140 billion in assets and that currently ranks as the world's second largest pension fund, assigns total risk limits to its active managers.
11. A general quartic equation (also called a "biquadratic equation") is a fourth-order polynomial of the form: $z^4 + a_3z^3 + a_2z^2 + a_1z + a_0 = 0$.

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