Nonlinear Term Structure Dependence: Copula Functions, Empirics, and Risk Implications

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Abstract

This paper documents nonlinear cross-sectional dependence in the term structure of U.S. Treasury yields and points out risk management implications. The analysis is based on a Kalman filter estimation of a two-factor affine model which specifies the yield curve dynamics. We then apply a broad class of copula functions for modeling dependence in factors spanning the yield curve. Our sample of monthly yields in the 1982 to 2001 period provides evidence of upper tail dependence in yield innovations; i.e., large positive interest rate shocks tend to occur under increased dependence. In contrast, the best fitting copula model coincides with zero lower tail dependence. This asymmetry has substantial risk management implications. We give an example in estimating bond portfolio loss quantiles and report the biases which result from an application of the normal dependence model.

Key words: affine term structure models, nonlinear dependence, copula functions, tail dependence, value-at-risk

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1 Introduction

The class of affine term structure models (ATSMs) as proposed by Duffie and Kan (1996) and further characterized by Dai and Singleton (2000), has recently become a benchmark in modeling the term structure of default-free interest rates. Within the model class, the term structure is characterized by the current realizations as well as the dynamics of a set of state variables. Only an exact assessment of the state variable dynamics and their dependence allows for an accurate description of the term structure. While previous empirical studies have focused on time-series nonlinearities and discontinuities in the process dynamics, this paper analyzes nonlinear cross-sectional dependence between factors that span the yield curve. We show that the dependence structure of the long and short end of the yield curve exhibits nonlinearity which can be characterized under a particular focus on extreme dependence. The implications of these findings are demonstrated by a risk management application.

Within the class of ATSMs, logarithmic bond prices are affine functions of the state variables. The class appeals by its analytical tractability and contains the well-known models by Vasicek (1977), Cox et al. (1985), Chen and Scott (1992), and Longstaff and Schwartz (1992), for example. However, recent empirical evidence indicates that term structure data do not support the ATSM class. A series of articles document distinct nonlinearities, e.g. in the drift and volatility function of the short-rate, which implies that mean-reversion in the short-rate depends on its level (see, for example, Aït-Sahalia (1996) and Stanton (1997)). Ang and Bekaert (2000) focus on these findings and develop a Markovian switching-model which captures such nonlinearities. Also, empirical results on one- and two-factor ATSMs by Duan and Simonato (1999) indicate a strong rejection of the affine model assumption when testing against alternatives which relax the restrictions imposed by the theoretical model.[1] In general, findings of nonlinearity in the term structure of interest rates are important for at least three reasons. First, only an exact assessment of the state variable dynamics and their dependence allows for an accurate modeling of the term structure. Second, derivatives pricing is frequently based on assumptions imposed by the class of ATSMs. Finally, effective bond portfolio risk management builds upon models which provide reliable risk implications.

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[1]Other specifications of the term structure include Ahn et al. (2002) as well as alternative formulations of the short rate e.g. by Chan et al. (1992) and Aït-Sahalia (1996). For an extensive survey of models see also Dai and Singleton (2002). Besides the linear structure, the distributional assumptions imposed by ATSMs is critical: Björk et al. (1997) extend the diffusion driven ATSMs by allowing for jumps. Eberlein and Raible (1999) study term structure models driven by general Lévy processes.

This paper analyzes nonlinear cross-sectional dependence between factors that span the yield curve. Performing a detailed statistical analysis, we document that the dependence structure between the short and the long end of the yield curve exhibits nonlinearity and we thereby put the focus on extreme dependence behavior. The starting point of our analysis is the benchmark-class of ATSMs. Based on this theory, the term structure dynamics in our study are given by a Gaussian two-factor generalized Vasicek model. This model was applied, for example, by Babbs and Nowman (1998) who find that the two-factor approach provides a good description of the yield curves for a broad sample of mature bond markets. Furthermore, Babbs and Nowman (1999) even support a two-factor specification as compared to a three-factor specification. They report that —although in formal statistical terms—the two factor model is rejected with the three-factor model as alternative, the measurement error of the two-factor model indicates that it performs as well as the three-factor model.

Formulating a discrete time two-factor model in state-space representation allows us to infer the parameters of the unobservable state variable process. As outlined in Duan and Simonato (1999), we use Lagrange multiplier statistics in order to test the validity of the Gaussian two-factor ATSM for U.S.-Treasury bond market data. A rejection of the model is followed by a statistical analysis based on copula functions without having to rely on Gaussian ATSM assumptions. We focus on cross-sectional dependence in the term structure and model general forms of dependence in the factor innovations. While a Gaussian factor model allows for correlated factors only, copula functions generalize the dependence concept; see e.g. Joe (1997) and Nelsen (1999). Dependence in the center of the distribution may be treated separately from the dependence in the distribution tails. Hence, we impose various combinations of symmetric as well as asymmetric tail dependence on the factor innovations.

Recent studies which apply copula functions in finance such as, for example, Ané and Kharoubi (2001) and Scaillet (2002) indicate that the concept is an appealing way of modeling complex dependence structures. Malevergne and Sornette (2002) argue that the hypothesis of the normal copula cannot be rejected for a variety of financial returns, including stock and exchange rate returns. However, this finding may relate to the amount of data available and to issues of power of the testing procedures in the presence of tail dependence. Indeed, the authors also find that alternative copula models cannot be rejected either. Conditional copula functions are studied by Patton (2001) who models conditional dependence in U.S.-Dollar exchange rate returns and by Rockinger and Jondeau (2001) who examine conditional dependence in international stock market returns. Within this literature, there is still debate on which copula models are most appropriate. We apply a

new transformed version of the Frank copula. Also, to our knowledge, the present study is first to provide evidence on the dependence structure within the term structure of interest rates.

In our empirical investigation, we study the term structure of U.S.-Treasuries which represent the largest government bond market worldwide. We use a sample of monthly yield curve observations as in the empirical studies for example by Ang and Bekaert (2000) and De Jong (2000). The sample covers the 20-year period from October 1982 to December 2001. We form two 10-year subsamples in order to check for the robustness of the empirical results. The empirical investigation in the paper is then organized in two steps. In the first step, we use the class of affine term structure models to specify the yield curve dynamics. In particular, we choose a two-factor generalized Vasicek model characterized by a jointly normal bivariate factor process. We then extract factors representing yields, namely the interest rates on zero-coupon bonds with one year and five years to maturity. The model parameters are estimated by Kalman filtering as supported by maximum likelihood arguments following the approach in previous studies such as, for example, Lund (1997), Duan and Simonato (1999), and Dewachter et. al. (2002). We test for alternatives by relaxing the restriction placed on the system by the affine term structure model as outlined by Duan and Simonato (1999). A rejection of the model is followed by further statistical analysis. In the second step, we model the dependence structure within the yield curve. We thereby focus on the dependence relation between short-term and long-term interest rates as represented by the two yield factors. To this aim, a broad set of different copula functions is used.

Based on our empirical findings, we show that the class of elliptical copula functions— including symmetric copulas such as the normal and the Student-t —has characteristics which violate the observed complex dependence structure. The yield factor dependence can neither be characterized by a correlation coefficient as in the normal model nor with a symmetric Student-t model. While the copula function of the normal distribution does not allow for dependence in the tails, the Student-t copula does not allow for asymmetric tail dependence. However, dependence models contained in the class of Archimedean copulas can indeed capture dependence in the yield curve which is characterized by distinct asymmetry and upper tail dependence. The Gumbel as well as a transformed Frank copula turn out to be more suitable choices than the student-t copula. Considering all candidate models used in our study, we find the transformed Frank copula to be the most appropriate model. Moreover, the goodness-of-fit tests for the two subsamples indicate that our main conclusions are robust within the observation period. In summary, using various different test methodologies, we conclude that the transformed Frank copula offers dependence

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characteristics which are more appropriate for the term structure innovations than those of other, commonly used, copula functions.

Given our empirical findings, we demonstrate the risk management implications in a bond portfolio setting. Based on the affine model of factor dynamics and the alternative copula models of factor dependence, we study the pricing effects of nonlinear dependence in the yield factors. Particularly, we use the ATSM implication that bond prices are exponential affine functions of the state variables. By sampling from one year and five year yield factors under the fitted copula functions we then estimate loss quantiles for bond portfolios with alternative durations. Our analysis highlights that the normal copula function—which is implied by the assumption of linear dependence in affine term structures—yields a substantial bias in the assessment of portfolio risk. When compared to the transformed Frank copula which captures the asymmetric dependence in the data, we report a bias structure in the upper and lower bond portfolio loss quantiles which yields values as high as six percent as compared to the normal model. The model bias due to the normality assumption should be even more pronounced when the pricing implications for nonlinear contracts, e.g. for interest rate derivatives, are considered.

The remainder of this paper is organized as follows. In the next section, we outline the model used in the analysis. Term structure time series dynamics are given with the class of ATSMs. Cross-sectional dependence in bivariate term structure innovations is modeled by two candidate classes of copula functions. The empirical investigation and the estimation results are given in Section 3. The application to bond portfolio risk management which points out risk implications of nonlinear factor dependence is given in Section 4. Section 5 concludes.

2   The Term Structure Model

The starting point of our study is the class of benchmark ATSMs. We model the term structure time series dynamics within a continuous time two-factor generalized Vasicek model. A state-space representation allows for observational noise and prepares estimation based on a discrete time vector autoregressive specification. We then focus on cross-sectional dependence in the term structure by modeling dependence in factor innovations with copula functions which stem from two broad dependence classes. Based on the copula model, we can impose various combinations of symmetric as well as asymmetric tail dependence on the factor innovations.

2.1 Term Structure Dynamics

2.1.1 Affine Term Structure Models

The affine term structure model is a class of models in which the yields to maturity are affine functions of some state variable vector. The state vector X is assumed to obey the following dynamics

$$dX(t) = \kappa(\theta - X(t))\,dt + \Sigma\sqrt{S(t)}\,dW(t), \tag{1}$$

where W is a d-dimensional standard Brownian motion, θ is a d-vector, κ and Σ are $d \times d$ matrices, and $S(t)$ is a $d \times d$ diagonal matrix with diagonal elements which are affine functions of the state vector X. Provided that a parameterization is admissible, the price of a zero bond $P(t, \tau)$ in time t with time to maturity τ can be expressed as

$$P(t, \tau) = \exp\left(A(\tau) + B(\tau)^\top X(t)\right), \tag{2}$$

where A is a scalar function, and B is a d-dimensional vector function. The instantaneous interest rate is, as usual, defined as

$$r(t) = -\lim_{\tau \searrow 0} \frac{\ln P(t, \tau)}{\tau}. \tag{3}$$

Duffie and Kan (1996) show that $P(\cdot, \cdot)$ is generically exponential affine, i.e. in the form of equation (2), if and only if the mean and variance in equation (1), and the short rate r are affine functions in the state variable X. Moreover, A and B in equation (2) are obtained as solutions to ordinary differential equations, see Duffie and Kan (1996). Let $R(t, \tau)$ denote the time-t continuously compounded yield on a zero bond with maturity τ. The yield to maturity of this bond is

$$R(t, \tau) = -\frac{\ln P(t, \tau)}{\tau}. \tag{4}$$

2.1.2 The Gaussian Two-Factor Model

The special case of the two-factor generalized Vasicek model is given by

$$\begin{aligned} r(t) &= R_0 + X_1(t) + X_2(t), \\ dX(t) &= -\kappa\, X(t)\,dt + \Sigma\, dW(t), \end{aligned} \tag{5}$$

where W is a 2-dimensional standard Brownian motion, and

$$\kappa = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \text{ and } \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ \rho\,\sigma_2 & \sqrt{1 - \rho^2}\,\sigma_2 \end{pmatrix}.$$

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The parameter R_0 is the mean level of the instantaneous rate r, the state processes X_1 and X_2 fluctuate around zero with mean reversion rates κ_1, κ_2, and diffusion coefficients σ_1, σ_2, and correlation ρ. Details on the functions A and B describing the term structure implied by the two-factor model are given in Duffie and Kan (1996) and Babbs and Nowman (1998). These functions are given by the factor parameters defined above and by γ_1 and $\gamma_2 \in \mathbb{R}$, which represent the risk premia of factor one and factor two, respectively.

2.1.3   State-Space Representation

Estimation of the above term structure model can be carried out via transformation to state-space representation; see for example Babbs and Nowman (1998, 1999) and Duan and Simonato (1999) for term structure estimation applications and Harvey (1989) for a general treatment of state-space models.

Assume that the yields for different maturities are observed with error. After the addition of measurement error, the yield to maturity, using the bond pricing formula (2), can be written as

$$R(t, \tau) = \frac{-A(\tau)}{\tau} + \frac{-B(\tau)^\top X(t)}{\tau} + \varepsilon(t, \tau), \tag{6}$$

where $\varepsilon(t, \tau)$ is assumed to be a normally distributed error term with mean zero and standard deviation $\sigma_{\varepsilon_\tau}$. Hence, given that N bond yields for different maturities are observed, the N corresponding yields have the following representation:

$$\begin{bmatrix} R(t, \tau_1) \\ \vdots \\ R(t, \tau_N) \end{bmatrix} = \begin{bmatrix} \frac{-A(\tau)}{\tau_1} \\ \vdots \\ \frac{-A(\tau)}{\tau_N} \end{bmatrix} + \begin{bmatrix} \frac{-B(\tau)^\top}{\tau_1} \\ \vdots \\ \frac{-B(\tau)^\top}{\tau_N} \end{bmatrix} X(t) + \begin{bmatrix} \varepsilon(t, \tau_1) \\ \vdots \\ \varepsilon(t, \tau_N) \end{bmatrix}. \tag{7}$$

In terms of the state-space model, this equation is referred to as the measurement equation. To obtain the transition equation for the state-space model, the expressions for the conditional mean and variance for the unobserved state variable process over a discrete time interval of length h have to be derived. Define $m(X(t); h) = \mathbb{E}\{X(t+h) \,|\, X(t)\}$ and $\Phi(X(t); h) = \text{Var}(X(t+h) \,|\, X(t))$, then the transition equation reads

$$X(t+h) = m(X(t); h) + \Phi(X(t); h)^{1/2} \eta(t, h), \tag{8}$$

where $\eta(t, h)$ is a d-vector of Gaussian white noise with $\Phi(X(t); h)^{1/2}$ denoting the Cholesky decomposition of $\Phi(X(t); h)$.

The two-factor model (5) defines the state variables as Gauss-Markov processes and thus the conditional mean and the conditional variance are:

$$m(x; h) = (m_{i,j}^h)_{2\times2} \cdot x = \begin{pmatrix} e^{-\kappa_1 h} & 0 \\ 0 & e^{-\kappa_2 h} \end{pmatrix} x, \tag{9}$$

$$\Phi(x; h) = (\Phi_{i,j}^h)_{2\times2} = \begin{pmatrix} \frac{\sigma_1^2}{2\kappa_1}(1 - e^{-2\kappa_1 h}) & \frac{\rho\sigma_1\sigma_2}{\kappa_1+\kappa_2}(1 - e^{-(\kappa_1+\kappa_2)h}) \\ \frac{\rho\sigma_1\sigma_2}{\kappa_1+\kappa_2}(1 - e^{-(\kappa_1+\kappa_2)h}) & \frac{\sigma_2^2}{2\kappa_2}(1 - e^{-2\kappa_2 h}) \end{pmatrix}. \tag{10}$$

Given observations of the yield vector in (6) and under a discrete sampling scheme with interval h, the exact likelihood function can be established based on the Kalman filter estimate of the unobservable state variable process X.

2.2 Nonlinear Term Structure Dependence

2.2.1 The Discrete-Time Factor Process

Section 2.1 above outlined the two-factor affine term structure model which we apply in our study for capturing the term structure dynamics. The generalized Vasicek model (5) is based on the continuous time factor dynamics dX driven by two-dimensional Brownian motion.

The factor process given by transition equation (8) is linear in the drift and non-stochastic in the diffusion coefficient. Hence, given (9) and (10), a discrete-time sample of X under $h = 1$, dropping h superscripts, is given by a vector autoregressive process of order one

$$\begin{aligned} X_{1,t} &= m_{1,1} X_{1,t-1} + (\Phi^{1/2})_{1,1}\eta_{1,t} + (\Phi^{1/2})_{1,2}\eta_{2,t}, \\ X_{2,t} &= m_{2,2} X_{2,t-1} + (\Phi^{1/2})_{2,1}\eta_{1,t} + (\Phi^{1/2})_{2,2}\eta_{2,t}, \end{aligned} \tag{11}$$

with $t = 1, ..., T$. The variables $\eta_{i,t}$, $i = 1, 2$ are uncorrelated iid standard normal innovations. In this setting, factor dependence is completely characterized by the correlation coefficient ρ. Generalizing the above model, we now rewrite the discrete-time factor dynamics as

$$\begin{aligned} X_{1,t} &= m_{1,1} X_{1,t-1} + Z_{1,t}, \\ X_{2,t} &= m_{2,2} X_{2,t-1} + Z_{2,t}, \end{aligned} \tag{12}$$

and assume that the innovations $(Z_{1,t}, Z_{2,t})$ are i.i.d. vectors with common joint distribution function $H(z_1, z_2)$. This relaxes the assumption of joint normality as imposed by the

class of ATSMs. Note that once the normality assumption does not hold, the affine structure essentially breaks down since it is derived on the basis of a multivariate Brownian motion as driving process and Itô calculus, see Duffie and Kan (1996).

2.2.2   Copula Functions

Based on (12), copula functions allow us to treat general versions of factor dependence in the two-factor generalized Vasicek model (5). The copula concept dates back to seminal papers by Hoeffding and Sklar; recent methodological overviews are given for example by Joe (1997) and Nelsen (1999). For the present application, we restrict the exposition to the two-dimensional case.

Let F_{Z_1} and F_{Z_2} denote the continuous marginal distribution functions of Z_1 and Z_2, i.e. $H(z_1, \infty)$ and $H(\infty, z_2)$, respectively. By transformation we obtain uniform random variables as $U = F_{Z_1}(Z_1)$ and $V = F_{Z_2}(Z_2)$. The copula function $C : [0,1]^2 \to [0,1]$ for the bivariate random vector (Z_1, Z_2) is defined as the joint distribution function of the uniform random vector $(U, V) = (F_{Z_1}(Z_1), F_{Z_2}(Z_2))$, that is, $C(u, v) = \mathbb{P}[U \leq u, V \leq v]$. Hence, it follows

$$H(z_1, z_2) = C(F_{Z_1}(z_1), F_{Z_2}(z_2)), \tag{13}$$

which is known as Sklar's Theorem. The result generally implies that for multivariate distribution functions the univariate margins and the dependence structure can be separated. Given that the marginal distribution functions are continuous, dependence is represented by a unique copula function C.

Apart from a separate treatment of dependence and marginal behavior, different copula functions may characterize dependence in the center of the distribution differently while showing identical limiting properties which characterize dependence in the distribution tails. Given the stylized fact of fat-tails in financial return distributions, tail dependence is therefore an interesting characteristic of copula functions. One can distinguish lower and upper tail dependence as defined below.

Definition 2.1 *The copula function C is* lower tail dependent *if*

$$\lim_{u \to 0} \frac{\mathbb{P}[U \leq u, V \leq u]}{u} = \lim_{u \to 0} \frac{C(u, u)}{u} = \lambda_L, \quad \lambda_L \in (0, 1],$$

and C is upper tail dependent *if*

$$\lim_{u \to 1} \frac{\mathbb{P}[U > u, V > u]}{1 - u} = \lim_{u \to 1} \frac{1 - u - u + C(u, u)}{1 - u} = \lambda_U, \quad \lambda_U \in (0, 1].$$

Since the tail dependence measures λ_L and λ_U are limit properties of a copula, we can write $\lambda_L = \lambda_L(C)$ ($\lambda_U = \lambda_U(C)$) or $\lambda_L = \lambda_L(\theta)$ ($\lambda_U = \lambda_U(\theta)$) if C is member of a parametric family C_θ with parameter vector θ. For the sake of simplicity, we may write λ whenever $\lambda_L = \lambda_U$.[2] We next introduce and characterize two standard classes of copula functions.

a) Elliptical copulas

The class of the elliptical copulas is widely used as a benchmark model. Elliptical copulas are commonly defined as copulas of elliptical distributions. In particular, this includes the copula of the student-t and the normal distribution function.

(a) The t-copula C_t is given by

$$C_t(u,v) = T_{\nu,\rho}\left(T_\nu^\leftarrow(u), T_\nu^\leftarrow(v)\right), \tag{14}$$

where $T_{\nu,\rho}$ is the bivariate standardized student-t distribution function with ν degrees of freedom and correlation ρ, while T_ν denotes the univariate standardized student-t distribution function and '$\leftarrow$' denotes the generalized inverse distribution function. The upper and lower tail dependence parameter λ for $\nu > 2$ is:[3]

$$\lambda = 2\left(1 - T_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)\right). \tag{15}$$

(b) For $\nu \to \infty$ the t-copula degenerates to the copula of the normal distribution

$$C_N(u,v) = N_\rho\left(N^\leftarrow(u), N^\leftarrow(v)\right), \tag{16}$$

where $N_\rho(\cdot)$ and $N(\cdot)$ denote the standard bivariate and the standard univariate normal distribution functions, respectively. From equation (15), it is obvious that zero tail dependence, i.e. $\lambda = 0$, results.

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[2]Note that in principle one cannot infer from a finite sample observation whether the underlying copula function is tail dependent or not. However, recent empirical studies, e.g. by Ané and Kharoubi (2001) and Malevergne and Sornette (2002), exhibit that the concept of tail dependence is useful in describing dependence structures in financial data.

[3]See Embrechts et al. (2002). Note that the above expression for λ even holds in the case $0 < \nu \leq 2$ then with a different interpretation of ρ.

b) Archimedean copulas

Elliptical copulas as outlined above are restricted to symmetry. For this reason, we outline the more general class of Archimedean copulas. They are described by a generator function φ as given in the proposition below.

Proposition 2.2 *Let* $\varphi : [0, 1] \rightarrow [0, \infty]$ *be continuous and strictly decreasing with* $\varphi(1) = 0$. *The function* $C : [0, 1]^2 \rightarrow [0, 1]$ *given by*

$$C(u, v) = \varphi^{[-1]} \left(\varphi(u) + \varphi(v) \right) \tag{17}$$

is a copula if and only if φ *is convex.*

Here $\varphi^{[-1]} : [0, \infty] \rightarrow [0, 1]$ denotes the pseudo-inverse of φ. The copula constructed by (17) is called Archimedean. The function φ is called generator of the copula. A generator φ is called strict if $\varphi(0) = \infty$ and in this case $\varphi^{[-1]} = \varphi^{-1}$. The following Archimedean copulas are utilized in this paper.

(a) The independence copula with generator $\varphi_\Pi(q) = -\ln q$ and

$$C_\Pi(u, \nu) = u\nu. \tag{18}$$

The copula exhibits neither lower nor upper tail dependence, i.e.: $\lambda_L = \lambda_U = 0$.

(b) The Gumbel copula with generator $\varphi_G(q) = (-\ln q)^\delta$ where $\delta \in [1, \infty)$ and

$$C_G(u, v) = \exp\left(-\left[(-\ln u)^\delta + (-\ln v)^\delta \right]^{\frac{1}{\delta}} \right). \tag{19}$$

It exhibits asymmetric tail dependence with zero lower tail dependence $\lambda_L = 0$ and upper tail dependence $\lambda_U = 2 - 2^{\frac{1}{\delta}}$. Note that overall dependence can be modeled only if upper tail dependence is non-zero, i.e. if $\delta > 1$.

(c) The Frank copula with generator $\varphi_F(q) = -\ln \frac{e^{-\vartheta \cdot q} - 1}{e^{-\vartheta} - 1}$ where $\vartheta \in (-\infty, \infty) \setminus \{0\}$ and

$$C_F(u, v) = -\frac{1}{\vartheta} \ln\left(1 + \frac{\left(e^{-\vartheta u} - 1 \right) \left(e^{-\vartheta v} - 1 \right)}{e^{-\vartheta} - 1} \right). \tag{20}$$

This copula is neither lower nor upper tail dependent, i.e. as for the independence copula we have: $\lambda_L = \lambda_U = 0$.

(d) In order to broaden the class of copula functions which may proof to be suitable for our modeling needs, we use a transformation rule as introduced by Nelsen (1999). The rule states that if φ is a generator and $\delta \geq 1$, then $\varphi_\delta(q) = \varphi(q)^\delta$ is also a generator.[4] Once we apply the transformation rule to the Frank copula C_F, the transformed Frank copula C_{TF} has generator $\varphi_{TF} = (\varphi_F)^\delta$ where the parameter vector is $\omega = (\vartheta, \delta) \in (-\infty, \infty) \setminus \{0\} \times [1, \infty)$. It follows that C_{TF} is given by:

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\begin{aligned}
C_{TF}(u, v) &= \varphi_{TF}^{-1}(\varphi_{TF}(u) + \varphi_{TF}(v)) \\
&= -\tfrac{1}{\vartheta} \ln \left[1 + \left(e^{-\vartheta} - 1\right) \exp \left[-\left(\left(-\ln \left[\tfrac{e^{-\vartheta \cdot u} - 1}{e^{-\vartheta} - 1} \right] \right)^\delta + \left(-\ln \left[\tfrac{e^{-\vartheta \cdot v} - 1}{e^{-\vartheta} - 1} \right] \right)^\delta \right)^{\frac{1}{\delta}} \right] \right].
\end{aligned}
$$
(21)

It can be shown that the transformed Frank copula C_{TF} has zero lower tail dependence, $\lambda_L = 0$, while it is upper tail dependent with $\lambda_U = 2 - 2^{\frac{1}{\delta}}$; in contrast to the Gumbel copula C_G defined in (19), it allows for overall dependence even if upper tail dependence is zero with $\delta = 1$ where it follows $C_{TF} = C_F$. Also, C_{TF} converges to the Gumbel copula C_G for $\vartheta \to 0$; see Junker and May (2002) for details.

3 Empirical Analysis of Nonlinear Term Structure Dependence

Our program for the empirical analysis is as follows. First we briefly introduce the zero-coupon yield dataset. Then we estimate the term structure parameters of the two-factor model based on a Kalman filter approach as outlined in Section 2.1.2. The empirical analysis of dependence between unpredictable innovations in the long end and the short end of the yield curve is based on an examination of our different parametric copula functions as given in Section 2.2.2. We argue that the theoretical properties of the copula functions given above, jointly with careful empirical testing, allow us to identify a suitable model which is consistent with the dependence in the yield structure.

3.1 The Sample

As pointed out in the introduction, our empirical analysis of the U.S.-Treasury term structure is based on a sample of monthly zero-coupon yields. The yield observations are obtained from the refined Fama-Bliss zero-coupon yield dataset as introduced in Fama and

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[4]Hence, the Gumbel copula C_G for example, follows immediately from the independence copula C_Π.

Bliss (1987). The maturities range from one to five years. The sample covers the period October 1982 to December 2001 with 231 monthly observations. Of course, the amount of sample information comes with a trade-off concerning stationarity. We did therefore not extend the sample back to periods in the early eighties where much different economic as well as economic policy regimes prevailed. Still, with the given dataset covering nearly twenty years, a check of robustness of the empirical results with respect to sample choice is important. We hence form two subsamples covering the October 1982 to December 1991 and the January 1992 to December 2001 period, which yields 111 and 120 monthly observations, respectively.

In the following, we consider the monthly zero-coupon yields $R(t, \tau_i)$, $t = 0$, ..., 230 where τ_i denotes the i-year maturity, $i = 1$, 2, 3, 4, 5. All yields are given on an annualized continuously compounded basis. The length of the discrete sampling interval, h, equals 1 month. Table 1 reports summary statistics for the entire sample period as well as for the two subperiods.

<center>(Table 1 about here)</center>

The statistics in Table 1 exhibit an on average increasing yield curve. The sample autocorrelation coefficients indicate typical first order linear dependence in monthly bond yields. Comparing the results for the subsamples with those for the entire period indicates lower levels of interest rates together with lower levels of volatility in yield changes for the second subsample.

3.2   Term Structure Model Estimation

This section presents the term structure estimation results based on Kalman filtering as outlined in Section 2.1.2. In general, the application of the Kalman filter requires the state process X to have normally distributed innovations which is typically violated in estimation applications. This implies that the parameter estimation approach is in effect based on a quasi-likelihood function.[5]

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[5]Inference based on the Kalman iteration and likelihood maximization faces two specification issues. First, the Kalman Filter estimates of X_t do not exactly correspond to the conditional expectations given the observed yields since the filter relies on a linear projection. Second, in a non-Gaussian model, the filtering errors—the differences between X_t and the linear projections—are not normally distributed. Brandt and He (2002) discuss the first-order approximation for non-normalities introduced by affine factor dynamics. Duan and Simonato (1999) discuss the estimation of square-root models by Kalman filtering and show in a simulation study that the biases are small, and, as also the results in Lund (1997) indicate, economically insignificant.

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Based on the Kalman filter approach, maximum likelihood (ML) estimation yields an estimate of the parameter vector $\psi = (R_0, \kappa_1, \kappa_2, \gamma_1, \gamma_2, \sigma_1, \sigma_2, \rho)$ of the two-factor generalized Vasicek model (5). We assume a diagonal covariance structure of the measurement errors $\varepsilon(t, \tau)$ in (7) where the diagonal elements are denoted by $\sigma_{\varepsilon_i}^2$. The estimation results are given in Table 2. All parameter estimates contained in ψ, apart from those of γ_1, turn out to be significantly different from zero at usual confidence levels. The estimated standard errors $\widehat{\sigma}_{\varepsilon_i}$ are relatively homogeneous for all maturities with a slight tendency for larger measurement error variability for the 1-year maturity yields. These results are are in line with those of previous empirical research. Additionally, we test for over-identification restrictions imposed by the model using a robust Lagrange multiplier test as suggested by Duan and Simonato (1999). The reported results clearly reject the Gaussian ATSM against the specified alternatives. This result is taken into account in the subsequent dependence analysis.

3.3 Derivation of Term Structure Innovations

Given the estimation results of the previous section, we can derive unpredictable innovations for our term structure sample. By choosing two observable yields, namely the short end τ_s-year and the long end τ_l-year maturity yield, $R(t, \tau_s)$ and $R(t, \tau_l)$, the dynamics of the two-dimensional yield factor X can be expressed in terms of the estimated term structure parameters; see the Appendix for details on the yield factor representation. For the Gaussian two-factor model it follows

$$
\begin{pmatrix} R(t, \tau_s) \\ R(t, \tau_l) \end{pmatrix} = \mu_R + A_R \begin{pmatrix} R(t-1, \tau_s) \\ R(t-1, \tau_l) \end{pmatrix} + \begin{pmatrix} \epsilon_{\tau_s, t} \\ \epsilon_{\tau_l, t} \end{pmatrix},
\tag{22}
$$

where μ_R and A_R are functions of the parameter vector ψ and $(\epsilon_{\tau_s, t} \epsilon_{\tau_l, t})^\top$ is a linear transformation of $(Z_{\tau_s, t} Z_{\tau_l, t})^\top$, i.e. $(\epsilon_{\tau_s, t} \epsilon_{\tau_l, t})^\top = \mathcal{B} (Z_{\tau_s, t} Z_{\tau_l, t})^\top$. Accordingly, the common joint distribution function $G(\epsilon_{\tau_s}, \epsilon_{\tau_l})$ of the innovations is completely determined by $H(z_{\tau_s}, z_{\tau_l})$ and $\mathcal{B}$ which is a function of ψ, see the Appendix for a derivation of $\mathcal{B}$.

In the following we choose the one-year and the five-year maturity yield, $R(t, 1)$ and $R(t, 5)$, to represent the short and the long yield factor, respectively. We present three alternative methods for deriving the bivariate empirical yield innovations $(\widehat{\epsilon}_{1, t}, \widehat{\epsilon}_{5, t})_{t=1, \ldots, T}$ which are then the basis of the subsequent copula analysis. Our first methodology is guided by a practitioner's viewpoint: we highlight the implications arising from the use of ATSMs.

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Thereby, the Kalman filter is understood as a tool for obtaining parameter estimates of the specific ATSM. Based on the estimates and the convenient affine structure of the model, the results may be applied to the valuation of interest rate options, risk management, and portfolio selection. Therefore, the measurement errors within the Kalman filter (see Equation 22), are not taken into account once the ATSM parameter estimates are obtained for the main part of the study. We refer to this approach as using the predictive affine system for deriving the empirical yield innovations. Based on the Kalman filter estimate $\widehat{\psi}$, we can derive the estimates $\widehat{\mu}_R$ and $\widehat{A}_R$. The time $t-1$ conditional expectation $\mathbb{E}_{t-1,\widehat{\psi}}$ is defined by equation (22). We then obtain the sequence of bivariate empirical yield innovations as

$$\begin{pmatrix} \widehat{\epsilon}_{1,t} \\ \widehat{\epsilon}_{5,t} \end{pmatrix} = \begin{pmatrix} R(t,1) - \mathbb{E}_{t-1,\widehat{\psi}}\{R(t,1)\} \\ R(t,5) - \mathbb{E}_{t-1,\widehat{\psi}}\{R(t,5)\} \end{pmatrix}, \quad t = 1, ..., T, \tag{23}$$

with $T = 230$.

The second methodology is related to the Lagrange multiplier test as outlined by Duan and Simonato (1996) and fully relaxes the ATSM assumption placed on the linear system given in (6). Based on the alternate specification, we obtain $\hat{\mu}_R^{\text{LM}}$ and $\hat{A}_R^{\text{LM}}$ and compute the empirical innovations $(\widehat{\epsilon}_{1,t}^{\text{LM}}, \widehat{\epsilon}_{5,t}^{\text{LM}})_{t=1,...,T}$ by

$$\begin{pmatrix} \widehat{\epsilon}_{1,t}^{\text{LM}} \\ \widehat{\epsilon}_{5,t}^{\text{LM}} \end{pmatrix} = \begin{pmatrix} R(t,1) - \mathbb{E}_{t-1}^{\text{LM}}\{R(t,1)\} \\ R(t,5) - \mathbb{E}_{t-1}^{\text{LM}}\{R(t,5)\} \end{pmatrix}, \quad t = 1, ..., T, \tag{24}$$

with $T = 230$, where $\mathbb{E}_{t-1}^{\text{LM}}$ is the time $t-1$ conditional expectation under the relaxed specification. This approach is referred to as using the the unrestricted system for deriving the empirical yield innovations.

We also provide a third methodology which incorporates the measurement errors of the Kalman filter in Equation (6). This approach is related to the work of De Rossi (2004) who also applies Kalman filtering on interest rate data. He utilizes the forecasted series $(\hat{R}_{t|t})_{t=1,...,T}$ for discussing the measurement error of the model. Here, we compute the so-called smoothed estimates $(\hat{X}_{t|T})_{t=1,...,T}$ of the states process X which are the estimates based on the entire sample information; see e.g. Harvey (1989). Employing the linear relation in (40), the smoothed yields $(\hat{R}_{t|T})_{t=1,...,T}$ are given by

$$\hat{R}_{t|T} = \mathcal{A} + \mathcal{B}\,\hat{X}_{t|T}\,. \tag{25}$$

This approach minimizes the effects of the measurement error and we obtain the most probable in sample estimate of the paths of the state process $(X_t)_{t=1,...,T}$ and of the observed yield system $(R)_{t=1,...,T}$ in the period of our study. We refer to this methodology

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as smoothed affine system. The empirical innovations $(\widehat{\epsilon}_{1,t}^{\text{SM}}, \widehat{\epsilon}_{5,t}^{\text{SM}})_{t=1,\dots,T}$ based on this method are given by

$$\begin{pmatrix} \widehat{\epsilon}_{1,t}^{\text{SM}} \\ \widehat{\epsilon}_{5,t}^{\text{SM}} \end{pmatrix} = \hat{R}_{t|T} - \left(\widehat{\mu}_R + \widehat{A}_R\, \hat{R}_{t-1|T} \right), \quad t = 1, \dots, T, \tag{26}$$

where $\widehat{\mu}_R$ and $\widehat{A}_R$ are given by the Kalman filter estimates $\widehat{\psi}$ which were obtained as a result of the first methodology.

The predictive affine system will be used for the main part of the analysis. Most results are also reported for the other methodologies. Note that including the estimation error of the Kalman filter estimate $\widehat{\psi}$ would be desirable. In this study, we have opted for leaving this issue out, and rather focus on the three different filtering methodologies. Based on the results, we conduct a detailed statistical analysis of the yield innovations' dependence structure.

3.4 Analysis of the Term Structure Innovations

Estimation of the copula parameter vector based on sample innovations is widely used in empirical research. Given the parameter estimates for the term structure dynamics above, the empirical marginal distribution functions are first determined for each component of the bivariate yield innovation series (23). We thereby investigate the joint normality assumption which is imposed by the ATSM. In the second step, parametric estimation of the copula functions is carried out. In order to avoid parametric model misspecification, we base our inference on the empirical marginal distributions and then derive the parameter estimates for the copula functions. The copula functions introduced in Section 2.2.2 are our respective candidate dependence models for the bivariate yield innovation series.

The Kalman filtering estimates of Section 3.2 result in the bivariate yield innovation series $(\widehat{\epsilon}_{1,t}, \widehat{\epsilon}_{5,t})_{t=1,\dots,T}$ as defined by equation (23). The following empirical results are restricted to the first methodology as introduced in Section 3.3. The analysis for the two alternative methodologies leads to rather similar results.

3.4.1 Univariate Distributional Properties of the Term Structure Innovations

The empirical marginal (univariate) distribution functions $F_{\cdot,T}$ are determined for each component separately. They are given as

$$F_{\cdot,T}(x) = \frac{1}{T} \sum_{t=1}^{T} I_{\{\cdot_t \leq x\}}. \tag{27}$$

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Closer inspection of the series' univariate distributional properties reveals that the assumption of uncorrelated normally distributed innovations requires more detailed inspection. Unreported QQ-plots of the marginal distributions of the innovations (where quantiles of the empirical distributions are plotted against those of the standard normal distribution) indicate a reasonably close approximation by the normal distribution; the fit in the lower tail is better for the long maturity factor innovations. Results from the univariate chi-square test indicate that normality cannot be rejected with p-values of 0.40 for $\widehat{\epsilon}_1$ and 0.15 for $\widehat{\epsilon}_5$. Anderson-Darling statistics for the univariate time series provide an alternative goodness-of-fit criterion. The in-sample statistics with values 0.160 for the one-year to maturity innovations and of 0.164 for the five-year to maturity innovations are both are well below 2.492 which represents the critical value for the 95% confidence level.

We next consider the innovations' time-series properties. Unreported sample autocorrelation functions for the univariate series with lags up to order 23 indicate that estimated autocorrelations stay within the 95% confidence intervals with one exception for $\widehat{\epsilon}_1$ which is an expected violation under the given confidence level. For the squared univariate innovations we find six violations which exceed the 95% confidence interval for the short as well as the long maturity series. The exceedances are evidence of some heteroskedasticity in both series. This finding is further supported by a comparison of the two subsamples which exhibit different levels of volatility (see the estimates of the standard deviation $\hat{\sigma}_1$ and $\hat{\sigma}_2$ in Table 4). The volatility level drops from approximately 0.0042 in the first period to approximately 0.0026 in the second period. [6] For previous findings on heteroskedasticity in bond yields refer to Bollerslev et al. (1992) and Bali (2000), for example.

Heteroskedasticity is frequently a matter related to estimation efficiency in empirical finance. In the following, we focus on nonlinear cross-sectional dependence. We thereby highlight a new empirical feature which violates the assumptions of the class of Gaussian ATSMs, while at the same time it is of significant economic relevance.

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[6]This effect leads to an inferior performance of out-of sample Anderson-Darling tests when compared to in-sample tests. Based on the out-of sample test performance, the null hypothesis is rejected in one out of two cases. We use the first subsample for estimating the copula parameters and apply these to compute the out-of sample statistics for the second subsample, and vice versa. The Anderson-Darling statistics increase significantly to 1.760 and 1.017 for the second subsample based on the first subsample's distribution estimates, and to 21.756 and 5.198 for the first subsample based on the second subsample's distribution estimates respectively.

3.4.2 Copula Estimation Methodology

Given the above results, we conclude that a most critical empirical issue of our term structure model is the dependence structure. As a valid simplifying prior, we assume in the following that the marginal distributions of the yield innovations are normal with mean zero and variances $\sigma_{1,\cdot}^2$ and $\sigma_{5,\cdot}^2$, respectively. Using Sklar's Theorem (13) we can write

$$G_{\cdot}(\epsilon_1, \epsilon_5) = C_{\cdot}(N(\epsilon_1/\sigma_{1,\cdot}), \ N(\epsilon_5/\sigma_{5,\cdot})), \tag{28}$$

where the notation ' $\cdot$ ' indicates the choice of one of the respective copula functions C_t, C_N, C_{TF} or C_G. For the different copula functions, the parameter vectors are given as $\omega_t = (\rho, \nu)$, $\omega_N = (\rho)$, $\omega_{TF} = (\vartheta, \delta)$, and $\omega_G = (\delta)$. Note that we use ML-estimation to obtain simultaneous estimates of the parameters $(\omega_{\cdot}, \sigma_{1,\cdot}, \sigma_{5,\cdot})$ of the joint distribution function $G_{\cdot}$. These estimates are optimal for the overall joint distributional assumption imposed by $G_{\cdot}$, which includes the marginal distributions as well as the dependence structure. With the joint density function $g_{\cdot}(\epsilon_1, \epsilon_5)$ derived from (28), the log likelihood function reads

$$\ln L_{\cdot}(\omega_{\cdot}, \sigma_{1,\cdot}, \sigma_{5,\cdot}; \ \epsilon_{1,t}, \epsilon_{5,t}) =$$

$$= \sum_{t=1}^{T} \ln \left[\frac{1}{\sigma_{1,\cdot}\sigma_{5,\cdot}} \ c_{\cdot}(N(\epsilon_{1,t}/\sigma_{1,\cdot}), \ N(\epsilon_{5,t}/\sigma_{5,\cdot})) \ N'(\epsilon_{1,t}/\sigma_{1,\cdot}) \ N'(\epsilon_{5,t}/\sigma_{5,\cdot}) \right], \tag{29}$$

where N' denotes the density of the standard normal distribution and $c_{\cdot}$ is one of our respective copula densities. Since the copula parameter estimates $\widehat{\omega}_T$ will have ML-properties, the estimates of the tail dependence parameters, $\widehat{\lambda}_T = \lambda(\widehat{\omega}_T)$ will be consistent and asymptotically normally distributed with

$$\sqrt{T}(\widehat{\lambda}_T - \lambda) \xrightarrow{d} N\left(0, \sigma_{\lambda}^2\right). \tag{30}$$

Given that $\lambda(\cdot)$ are suitably smooth functions, the variance term σ_{λ}^2 can be approximated by a first order Taylor series expansion of the form

$$\sigma_{\lambda}^2 = \left(\frac{\partial \lambda(\widehat{\omega}_T)}{\partial \omega_T}\right)^{\top} \Sigma_{\widehat{\omega}_T} \frac{\partial \lambda(\widehat{\omega}_T)}{\partial \omega_T},$$

where $\frac{\partial \lambda}{\partial \omega_T}$ denotes the gradient of the tail dependence as a function λ of the parameters ω_T, and $\Sigma_{\widehat{\omega}_T}$ denotes the covariance matrix of the parameters $\widehat{\omega}_T$ produced by the ML estimation.

3.4.3 Diagnostics for the Estimated Copula Models

Based on our estimates of the parametric copulas, we compare the in-sample model fit based on a set of different goodness-of-fit test procedures. These include seven test statistics which are given as follows.

A general goodness-of-fit test is the bivariate version of the well-known χ^2-test. Unfortunately, standard theory does not apply to empirical copula processes; see van der Vaart and Wellner (1996). We evaluate the goodness-of-fit of several parametric models for the joint distribution of the innovations $(\epsilon_{1,t}, \epsilon_{5,t})$. Hence, we perform a conventional χ^2-test on the original space $\mathbb{R}^2$ of the innovations. Therefore, $\mathbb{R}^2$ is divided into rectangles (c_i) forming a Donsker class, and we define the χ^2-statistic by

$$X_{\mathrm{df}}^2 = \sum_{i=1}^{k} \frac{(\mathbb{E}_{G.}(c_i) - \#\{(\epsilon_{1,t}, \epsilon_{5,t}),\ t = 1,...,T : (\epsilon_{1,t}, \epsilon_{5,t}) \in c_i\})^2}{\mathbb{E}_{G.}(c_i)}, \tag{31}$$

where $\mathbb{E}_{G.}(c_i)$ denotes the expected number of observations in cell c_i under the model $G..$. For the χ^2-test, separation of the cells (c_i) has to be carried out in a way as to get space resolvent test information while at the same time guarantee convergence of the empirical process. We follow Moore (1986) who suggests to choose $\mathbb{E}_{G.}(c_i) \approx T^{2/5}$ for every cell; accordingly 28 cells follow for the full sample and 17 cells follow for the two subsamples, respectively. Due to the restricted sample size in our setting, we follow Pollard (1979) for the asymptotic distribution of the statistic, X_{df}^2, based on the ungrouped estimator

$$X_{\mathrm{df}}^2 \to^d \chi_{\mathrm{df}-m-1}^2 + \sum_{l=1}^{m} \alpha_l\, \chi_1^2,$$

where df is the number of cells, m is the number of model parameters, and $(\alpha_l)_{l=1,...,m}$ are constants taking values in $[0, 1]$. Setting $\alpha_l = 0$ or $\alpha_l = 1$, we can derive upper and lower bounds for the p-values. Hence, we provide intervals for the p-values of the χ^2-statistic X_{df}^2.

Three additional tests of the overall model fit are based on the maximized log-likelihood function $\ln L.$. These include the Akaike information criterion, $AIC = -2\ln L. + 2p$, and the Bayesian information criterion, $BIC = -2\ln L. + p\ln T$. While the Bayesian criterion puts a heavier penalty on the number of model parameters, both statistics are based on the probability of the observations within a given model. In contrast to that, the entropy criterion measures the probability of a given model. The model entropy is given as the expected value of the negative logarithm of the maximized density function, $EN = \mathbb{E}(-\ln g.(\epsilon_1, \epsilon_5))$, where we approximate the expectation by Monte Carlo simulation.

The χ^2-tests can be interpreted as a measure of the differences in the densities. Alternative goodness of fit statistics in the literature, such as the Kolmogorov/Smirnov-test, are based on distance measures between the empirical and a theoretical parametric distribution function. As both, the empirical and the theoretical distribution functions have to converge to zero at the lower tail and to one at the upper tail, their representation of fit in the tails is weak by construction. A test statistic which is superior in this regard dates back to Anderson and Darling (1952). The Anderson-Darling test uses relative instead of absolute deviations between the distribution functions and thereby gives a better representation of the fit in the tails. We use the integrated version outlined by the authors and denote it by AD.

Our last goodness-of-fit diagnostic particularly focuses on the fit in the distribution tails. The AD statistic, due to its use of the cumulative distribution function, has the drawback of a smoothing effect which is particularly present in the upper tail. Considering model fit in the tails, we therefore also apply a diagnostic which is not based on the overall probability deviations, but on the probability deviations at a particular quantile of the joint distribution function only. Let C_T denote the empirical copula function

$$C_T(u,v) = \frac{1}{T} \# \{(u_t, v_t), \ t = 1, ..., T : (u_t, v_t) \leq (u, v)\}, \tag{32}$$

and $\overline{C}_T$ denote the empirical survival copula function

$$\overline{C}_T(u,v) = \frac{1}{T} \# \{(u_t, v_t), \ t = 1, ..., T : (u_t, v_t) > (u, v)\}. \tag{33}$$

With these empirical copulas we measure deviations at the upper and the lower tail independently. The relative lower tail probability deviation PD_p is defined as the deviation of the model probabilities from the empirical probabilities measured at a point (q, q) in the lower corner of the set $[0, 1]^2$. Here, $q = C_T^{\leftarrow}(p)$, $0 \leq p \leq 1$ and $C_T^{\leftarrow}$ is the inverse of the diagonal section of the empirical copula function C_T. The relative lower tail probability deviation PD_p is given as

$$PD_p = \frac{G. \left(F_{\epsilon_1,T}^{\leftarrow}(C_T^{\leftarrow}(p)), \ F_{\epsilon_5,T}^{\leftarrow}(C_T^{\leftarrow}(p))\right) - p}{p}. \tag{34}$$

Based on survival functions, the upper tail relative probability deviation is defined by the survival probability deviation $\overline{PD}_p$ at the point (q, q) in upper corner of $[0, 1]^2$. It is given as

$$\overline{PD}_p = \frac{\overline{G}. \left(\overline{F}_{\epsilon_1,T}^{\leftarrow}(\overline{C}_T^{\leftarrow}(p)), \ \overline{F}_{\epsilon_5,T}^{\leftarrow}(\overline{C}_T^{\leftarrow}(p))\right) - p}{p}. \tag{35}$$

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Setting p equal to a small positive value, the probability deviations PD_p and $\overline{PD}_p$ allow us to measure deviations in the tails. Note that pT observations are available for the calculation of the empirical distribution function and—given that pT is sufficiently large—ensure convergence towards the theoretical distribution function.

3.4.4 Copula Estimation Results

Referring back to the methodology introduced in Section 3.3, Tables 3 and 4 give the estimation results for the overall sample 1982-2001 and the two subsamples 1982-1991 and 1992-2001, respectively. Giving additional results, Table 5 shows the corresponding results for the overall sample 1982-2001 under the second and the third methodology, respectively. We give standard deviation estimates for the marginal distributions as well as the estimates of the parameters in the copula parameter vectors ω_t, ω_N, ω_{TF}, and ω_G. Further, we provide results of nested likelihood ratio tests of the normal copula against the t-copula and the Gumbel-copula against the transformed Frank-copula. Note that testing the normal-copula against the t-copula is a boundary value problem, since the degree of freedom of the t-copula approaches infinity in order to describe the null model which is the normal-copula. Following Self and Liang (1987), the distribution is not χ^2 but censored χ^2, i.e., the deviance d has the asymptotic distribution $N^2 \mathbf{1}_{\{N \geq 0\}}$ where N is a standard normal random variable. In Table 6 we compare the goodness-of-fit for the competing copula models for the overall sample as well as for the two subsamples. Table 7 provides these results for the second and third methodology as introduced in Section 3.3. For the evaluation we use the seven statistics χ^2, AIC, BIC, EN, AD, PD and $\overline{PD}$ as defined above in Section 3.4.3.

(Table 3 about here)

(Table 4 about here)

(Table 5 about here)

Table 3 summarizes the estimation results for the joint distribution functions in the overall sample period. The normal copula yields an estimate of the correlation coefficient of 0.85, which indicates a quite strong positive linear dependence in the yield factors. The subsample results in Table 4 indicate comparable linear dependence, with an estimate of 0.89 in the first and 0.79 in the second subsample. The results also show that the estimates of the standard deviations for the marginal distributions vary somewhat depending on the copula model, where the Gumbel copula assigns the largest standard deviations to the

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margins. Considering the subsamples, this highlights that the standard deviations in the yield factor realizations were higher in earlier subsample period 1982-1991 than in the later 1992-2001 period.

A graphical illustration of the estimated 1982-2001 joint distribution functions under the different copula functions is given in Figure 1. The plotted joint density contour lines visualize the dependence implications of the different copulas, while the plotted yield innovations allow for a first visual inspection of model fit.

(Figure 1 about here)

(Table 6 about here)

(Table 7 about here)

We next turn to the results of the goodness-of-fit tests in Tables 7 and 8. Given the size of the data set we have to point out in advance that it is not possible to strictly reject any of the copula models. However, it turns out that the transformed Frank copula shows best overall fit and also never obtains one of the worst diagnostic results in any of the two subsamples.

Starting with the overall sample, our results clearly indicate that the transformed Frank copula is the superior dependence model. All seven statistics including those which penalize for the number of model parameters (χ^2, AIC and BIC) happen to favor the C_{TF}-model where the second best model follows with some diagnostic distance. Considering the symmetric models, the student-t copula shows advantages in the chi-square and the entropy statistic, but not for AIC and BIC. It is remarkable that the AD test always shows very high deviations values for the student model as compared to the other models. Considering Figure 1, an explanation for this finding may be that that the contours of the student model narrow most quickly in the overall region of the lower left quadrant $[-\infty, 0] \times [-\infty, 0]$ of the joint distribution function thereby causing large relative deviations in the empirical versus theoretical distributions for moderate negative values. Additionally, due to the sample size, the AD statistic should be interpreted with some caution; the number of 230 observations may not fully guarantee convergence of the empirical distribution functions, which is a requirement for the AD statistic. Concerning the probability deviations in the tails, PD and $\overline{PD}$, we choose $p = 0.05$ which yields 11 observations for the calculation of the marginal distribution function. The PD and $\overline{PD}$ results indicate that the transformed Frank copula has lowest deviations from the empirical observations in the upper as well as the lower tail. The symmetric models tend

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to overestimate the probability of observations in the lower tail, which is demonstrated by large positive values for PD. The Gumbel copula shows a tendency to overestimate the probability of observations in the upper tail showing a large deviation $\overline{PD}$; note that the Gumbel copula models overall dependence and upper tail dependence jointly via the δ-parameter which implies strong upper tail dependence under strong linear dependence, and vice versa. Using likelihood ratio tests, we can compare the nested specifications, i.e. normal against student-t and Gumbel against transformed Frank. In both cases, the p-values indicate that the LR test favors the two-parameter alternatives against the single-parameter models. Hence, the dependence structure, whether symmetric or asymmetric, can hardly be handled by a single-parameter specification.

These results for the first methodology are consistent with those of the second and third methodology for obtaining the yield innovations; see Table 5 for the copula estimates, and Table 7 for the goodness-of-fit statistics. The unrestricted system has very similar estimates as compared to the predictive affine system. The smoothed affine system produces somewhat different copula estimation results. The correlation coefficient estimate for the normal and the student-t copula increases as well as the tail dependence for all copula specifications. This effect can be explained by the nature of the smoothed system, which reduces the impact of the measurement error on the observations and therefore, the dependence between the factors is more noticeable. Moreover, for both alternative methodologies the p-values of the LR tests decrease giving even stronger evidence for the t-copula against the normal-copula and the transformed Frank against the Gumbel copula.

We next turn to the subsamples, i.e. the 1982-1991 and 1992-2001 subsample results. Note that the PD and $\overline{PD}$ statistics are now based on $p = 0.1$ which, under a subsamples sizes of roughly $T/2$, implies a number of tail observations roughly equal to these for the full sample diagnostics. For the subsamples, the assignment of the best goodness-of-fit statistics in Table 6 shows notable variations across the models. This is due to the substantial decrease in sample size, which makes the interpretation of the results less conclusive than for the overall sample. However, the transformed Frank copula still obtains the best results when evaluated by a score of the number of best fit-results among all models. Also, the C_{TF}-copula function is never assigned one of the worst-fit results, which does not hold for any of the other models. Given a smaller data set, the statistics χ^2, AIC and BIC considerably penalize the two-parameter copula functions. At the same time, the C_t parameter estimate of ν for the first susample as well as the C_{TF} parameter estimate of δ for the second subsample (see Table 4), exhibit low respective t-values. A relatively stable pattern in Table 6 is provided by the $\overline{PD}$ statistics; high deviations for

the overall sample as well as for the subsamples point out that a drawback of the Gumbel copula is its tendency to overestimate the upper tail.

Turning to the first subsample, 1982-1991, C_{TF} shows the best fit according to the EN, AD and PD statistics. The one-parameter copula models C_N and C_G also perform relatively well. The normal model even gives best fit according to the chi-square test statistic and $\overline{PD}$, and the the Gumbel model produces the best results for AIC and BIC. In the second subsample, 1992-2001, as mentioned above, we report lower estimates of the volatility in the marginal distributions. However, our results do not indicate that the dependence structure is much different in the two subsamples. The C_{TF}-copula again yields results better than for the other models with best fit as measured by the χ^2-statistic, the BIC, the EN and the upper tail fit $\overline{PD}$. The student-t model has the worst AD statistic; still it has the best fit according to the AIC and PD measures. The normal copula performs notably well having the best AD statistic and none of the worst results. The LR test results for testing normal against student-t and Gumbel against transformed Frank are provided in Table 4. In the first subsample the null models are not rejected, where we have highly significant rejections of the null models for the second subsample. To summarize the subsample comparison results, we can state that – given a high variability in the statistics – the Gumbel copula provides a second best fit in the first subperiod while the normal copula provides a second best fit in the second subperiod. In both subperiods however, the statistics indicate that the transformed Frank copula has best overall fit. This result is also robust with respect to the three different methodologies applied for computing the empirical yield innovations.

4   Application: Measuring Bond Portfolio Risk

Based on the affine term structure model of Section 2.1.2, the term structure of interest rates is completely described by two risk factors. Clearly, the dependence characteristics of the joint distribution of the 1-year yield and the 5-year yield influences the risk measurement of portfolios. In this section we analyze the different distributional specifications' impact on risk management decisions. We apply the estimation results obtained under the predictive affine system which are summarized in Table 2.

We start with a graphical illustration of the copula function estimation results of Section 3. Figure 2 contains four plots of the fitted conditional densities of the 5-year yield given a fixed realization of the 1-year yield. Each plot represents one of the four different copula models. As can be seen, the conditional densities show large structural differences

especially including the probability of joint upper or lower tail events. For example, given a negative shock to the short rate of –0.02 in the transformed Frank model, the conditional density for the five year yield has high variance while, given a positive shock, the conditional density has low variance. As is illustrated, the structure looks much different for the symmetric models.

(Figure 2 about here)

We next apply the estimated dependence structures to quantify the risk of different bond portfolios and compare the results. We consider portfolios which invest in the 1-year zero bond at price $P(t, 1)$ and the 5-year zero bond at price $P(t, 5)$ and then study the return of this investment after 1 month of time. Denote by R_Λ the return of the portfolio which has initial portfolio duration Λ. As a risk measure ϱ we utilize Value-at-Risk (VaR), i.e. the quantile of the profit-and-loss distribution of the portfolio. When adjusted for the expected portfolio return VaR is

$$\varrho = F_{R_\Lambda}^{\leftarrow}(\alpha) - \mathbb{E}\{R_\Lambda\},$$

where α is the confidence level and F_{R_Λ} is the cumulative distribution function of R_Λ. We introduce the superscript '+' to the risk measure ϱ when measuring the risk of a long position, and the superscript '−' for measuring the risk of a short position, respectively. Additionally, the subscript at the risk measure ϱ indicates the copula applied for defining the dependence structure. The confidence levels we discuss are $\alpha = 99\%$ and $\alpha = 99.9\%$ for which the VaR numbers ϱ_N^+, ϱ_t^+, ϱ_G^+, ϱ_{TF}^+, and ϱ_N^-, ϱ_t^-, ϱ_G^-, ϱ_{TF}^- are calculated.

We compare the risk measures ϱ by fixing the risk measure induced by the normal copula which is the standard risk measure, and calculate the relative deviations from this measure. The relative deviations are

$$\Delta_. = \frac{\varrho_. - \varrho_N}{\varrho_N},$$

where '$\cdot$' indicates the choice of one of the three copula functions C_t, C_{TF}, and C_G. The relative deviations $\Delta_.$ for long and short bond portfolio holdings as a function of the initial duration Λ are plotted in Figure 3.

(Figure 3 about here)

The results for holding a long position in the interest rate portfolio Δ^+ are displayed at the top of Figure 3. The results for t-copula model Δ_t^+ are indicated by the solid line.

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As can be seen at the top left in Figure 3, the t-copula produces VaR numbers which are close to the normal copula model for the 99% confidence level. When increasing the confidence level to 99.9% at the top right in Figure 3, the maximum relative deviation increases from 0.2% to 3% which reflects the property of the t-copula to adopt to the (upper) tail dependence existing in our data set. A similar pattern is observed for the model given by transformed Frank copula (see the dashed line). The relative deviation Δ_{TF}^{+} takes a maximum value of approximately 4% for the 99.9% confidence level, and the VaR is persistently larger than the numbers based on the t-copula model. The Gumbel copula (dashed-dotted line) generates the highest VaR. The maximum relative deviation Δ_{G}^{+} is 2.5% for the 99% confidence level, and around 4.5% for the 99.9% confidence level, respectively.

At the bottom of Figure 3 the results Δ^{-} are shown for a short position in bond portfolios with initial duration Λ. The relative deviation of the t-copula model Δ_{t}^{-} has similar characteristics as in the case of the long position. The VaR turns out to be relatively close to the VaR given under the normal copula, where the positive deviations tend to become overall larger when the confidence level is increased from 99% to 99.9%. The t-copula quantiles exceed the normal ones because the t-copula features lower tail dependence which is not present in the data. In contrast to the t-copula, the transformed Frank copula and the Gumbel copula both produce negative relative deviations of the VaR measures when compared to the normal copula. The maximum relative deviation is around 3% for the 99% confidence level and around 6% for the 99.9% confidence level.

The above findings can be interpreted as follows. In Section 3.4.4, the transformed Frank copula proved to be the dependence model which reflects the observed dependence structure in the most appropriate way. Assuming that the data are generated by a joint distribution with normal margins and a transformed Frank copula then implies that the normal copula produces a systematic bias in measured VaR.

For long bond portfolio positions, the normal copula tends to underestimate VaR where the lack in risk capital may approximately amount to up to 4% in our example. Clearly, the negative bias in VaR produced by the normal copula is related to the upper tail dependence which is present in the data but not a characteristic of the normal dependence model. The t-copula results in VaR numbers which are much closer to the transformed Frank numbers than the normal numbers with a maximum deviation of approximately 1%. This finding is due to the upper tail dependence which is incorporated in the t-copula. The Gumbel copula features characteristics which are present in the analyzed data set: upper tail dependence and asymmetry. The VaR numbers are of reasonable

quality especially for the high confidence levels of 99.9%. For the 99% confidence level, the Gumbel copula produces the maximum relative deviation to transformed Frank numbers. For this particular case, the Gumbel copula performs poorly compared to the alternative dependence models (Figure 3, top, left).

For short bond portfolio positions, the normal copula overestimates VaR. The maximum relative deviation takes a value around 6%. Though the data were not found to show lower tail dependence in Section 3.4.4, which is in accordance with the normal copula, bias is again present having opposite sign. The explanation for this finding is reasonably simple. As the upper and lower tail of the normal copula are estimated simultaneously, the realized estimate is a balanced result of both shortcomings of the normal copula, namely its of lack of tail dependence and its symmetry characteristic. Also, due to its symmetric structure, the absolute biases generated by the t-copula are high when compared to the transformed Frank model. Hence, the t-copula turns out to produce overestimated VaR numbers for short positions. As it turns out, even for moderate confidence levels of 99%, the copula functions' ability to reproduce a complex observed dependence structure gains in importance. The Gumbel copula is a parsimoniously parameterized model which captures upper tail dependence. The relative deviations for the best-fitting transformed Frank model indicate that the normal model can produce VaR biases of up to 6% in the given example.

5 Conclusion

As is well-known, the concept of linear dependence breaks down under non-normality. The present investigation documents that some of the more flexible nonlinear dependence models derived form statistical theory may prove to be relevant in financial applications.

Based on the benchmark model given by the affine class of term structures which assumes joint normality in yield innovations, this paper analyses cross-sectional dependence in the term structure of U.S.-Treasury yields. The nonlinearities documented in the data represent a profound statistical characteristic which is shown to be of economic significance. Deviations from linear dependence have implications on risk management when financial risk is for example measured by the commonly used VaR methodology. Most strikingly, we conclude that the normal copula as a benchmark model of dependence imposes two main problems, namely absence of tail dependence and symmetry, which prevent accurate risk measurement. Our findings are not limited to bond pricing and bond portfolio VaR applications. The model bias due to the normality assumption should be even more

pronounced when the pricing implications for nonlinear contracts, e.g. for interest rate derivatives, are considered.

Acknowledgement

Appendix A. Yield-Factor Representation

Provided that a parameterization is admissible, we know from Duffie and Kan (1996):

$$P(t,\tau) = \exp\left(A(\tau) + B(\tau)^\top X(t)\right) , \tag{36}$$

where the functions A and B describing the two-factor generalized Vasicek model of Babbs and Nowman (1998) are given by the parameters $R_0 \in \mathbb{R}$, $\kappa_1, \kappa_2, \sigma_1, \sigma_2 \in \mathbb{R}^+$, $\rho \in]-1, 1[$, and $\gamma_1, \gamma_2 \in \mathbb{R}$ in the following way:

$$A(\tau) = \sum_{i=1,2} \left[-\pi_i \left(B_i(\tau) + \tau\right) - \frac{\sigma_i^2}{4\,\kappa_i} B_i(\tau)^2 \right] - R_0\tau + A_\rho(\tau), \tag{37}$$

$$B(\tau) = (B_1(\tau), B_2(\tau))^\top = \left(\frac{e^{-\kappa_1 \tau} - 1}{\kappa_1}, \frac{e^{-\kappa_2 \tau} - 1}{\kappa_2} \right)^\top , \tag{38}$$

where $\pi_1 = \frac{\sigma_1 \gamma_1}{\kappa_1} - \frac{\sigma_1^2}{2\,\kappa_1^2}$, $\pi_2 = \frac{\sigma_2 \left(\rho\,\gamma_1 + \sqrt{1-\rho^2}\,\gamma_2\right)}{\kappa_2} - \frac{\sigma_2^2}{2\,\kappa_2^2}$, and:

$$A_\rho(\tau) = \frac{\rho\,\sigma_1\,\sigma_2}{\kappa_1 + \kappa_2} \left[\frac{1}{\kappa_1}(B_1(\tau) + \tau) + \frac{1}{\kappa_2}(B_2(\tau) + \tau) - B_1(\tau)\,B_2(\tau) \right]. \tag{39}$$

A distinct feature of ATSM framework is that the latent state variables can be transferred to an appropriate set of yields, see Duffie and Kan (1996). Moreover, the affine structure

of the latent variables is preserved for the yields, and the yields can be viewed at as a new set of state variables provided some technical conditions hold. Given a d-factor ATSM with state variable $X = (X_1, ..., X_d)^\top$. For a set of maturities $(\tau_1, ..., \tau_d)$ the corresponding yields $Y = (Y_1, ..., Y_d)^\top$ are given by Equation (36):

$$Y(t) = \mathcal{A} + \mathcal{B} X(t) \tag{40}$$

where

$$\mathcal{A} = \begin{pmatrix} -\frac{A(\tau_1)}{\tau_1} \\ \vdots \\ -\frac{A(\tau_d)}{\tau_d} \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} -\frac{B_1(\tau_1)}{\tau_1} & \cdots & -\frac{B_d(\tau_1)}{\tau_1} \\ \vdots & \ddots & \vdots \\ -\frac{B_1(\tau_d)}{\tau_d} & \cdots & -\frac{B_d(\tau_d)}{\tau_d} \end{pmatrix}. \tag{41}$$

Provided B is non-singular, we can state the following equation for the yield vector Y:

$$dY(t) = \tilde{\kappa}(\tilde{\theta} - Y(t))\, dt + \widetilde{\Sigma}\sqrt{\widetilde{S}(t)}\, dW(t)\,, \tag{42}$$

where

$$\tilde{\kappa} = \mathcal{B}\,\kappa\,\mathcal{B}^{-1}\,, \quad \tilde{\theta} = \mathcal{B}\,\theta + \mathcal{A}\,, \quad \widetilde{\Sigma} = \mathcal{B}\,\Sigma\,, \text{ and } \widetilde{S}(t) = \{\tilde{\alpha}_i + \tilde{\beta}_i^\top Y(t)\}\,, \tag{43}$$

and $\tilde{\alpha}_i = \alpha_{S\,i} - \beta_{S\,i}^\top\, \mathcal{B}^{-1}\mathcal{A}$, and $\tilde{\beta}_i = \mathcal{B}^{-1^\top} \beta_{S\,i}$. We briefly discuss the yield dynamics implied by the two-factor generalized Gaussian model. Here, we find $\theta = 0$, and $S(t) = \mathbb{I}_d$ what results into $\tilde{\kappa} = \mathcal{B}\,\kappa\,\mathcal{B}^{-1}$, $\tilde{\theta} = \mathcal{A}$, $\widetilde{\Sigma} = \mathcal{B}\,\Sigma$, and $\widetilde{S}(t) = \mathbb{I}_d$:

$$dY(t) = \mathcal{B}\,\kappa\,\mathcal{B}^{-1}(\mathcal{A} - Y(t))\, dt + \mathcal{B}\,\Sigma\, dW(t)\,, \tag{44}$$

where $\mathcal{B}\,\kappa\,\mathcal{B}^{-1}$ describes the mean reversion including cross-dependencies between Y_1 and Y_2, and the covariance is given by $\mathcal{B}\,\Sigma\,\Sigma^\top\,\mathcal{B}^\top$. Setting $(R(\cdot, \tau_s), (R(\cdot, \tau_l))^\top = (Y_1, Y_2)$, it follows from the affine structure

$$\begin{pmatrix} R(t, \tau_s) \\ R(t, \tau_l) \end{pmatrix} = \mathcal{A} + \mathcal{B}X_t = \begin{pmatrix} -A(\tau_s)/\tau_s \\ -A(\tau_l)/\tau_l \end{pmatrix} + \begin{pmatrix} -B(\tau_s)^\top/\tau_s \\ -B(\tau_l)^\top/\tau_l \end{pmatrix} X_t.$$

Due to the autoregressive structure of X in (12) this results in

$$\begin{pmatrix} R(t, \tau_s) \\ R(t, \tau_l) \end{pmatrix} = (\mathbf{I}_2 - \mathcal{B}\,K\,\mathcal{B}^{-1})\,\mathcal{A} + \mathcal{B}\,K\,\mathcal{B}^{-1}\begin{pmatrix} R(t-1, \tau_s) \\ R(t-1, \tau_l) \end{pmatrix} + \mathcal{B}\begin{pmatrix} Z_{\tau_s, t} \\ Z_{\tau_l, t} \end{pmatrix}\,,$$

which is Equation (22) where $\mu_R = (\mathbf{I}_2 - \mathcal{B}\,K\,\mathcal{B}^{-1})\,\mathcal{A}$ and $A_R = \mathcal{B}\,K\,\mathcal{B}^{-1}$, and $K = \mathrm{diag}(e^{-\kappa_1 h}, e^{-\kappa_2 h})$, see Equation (9) and Equation (12).

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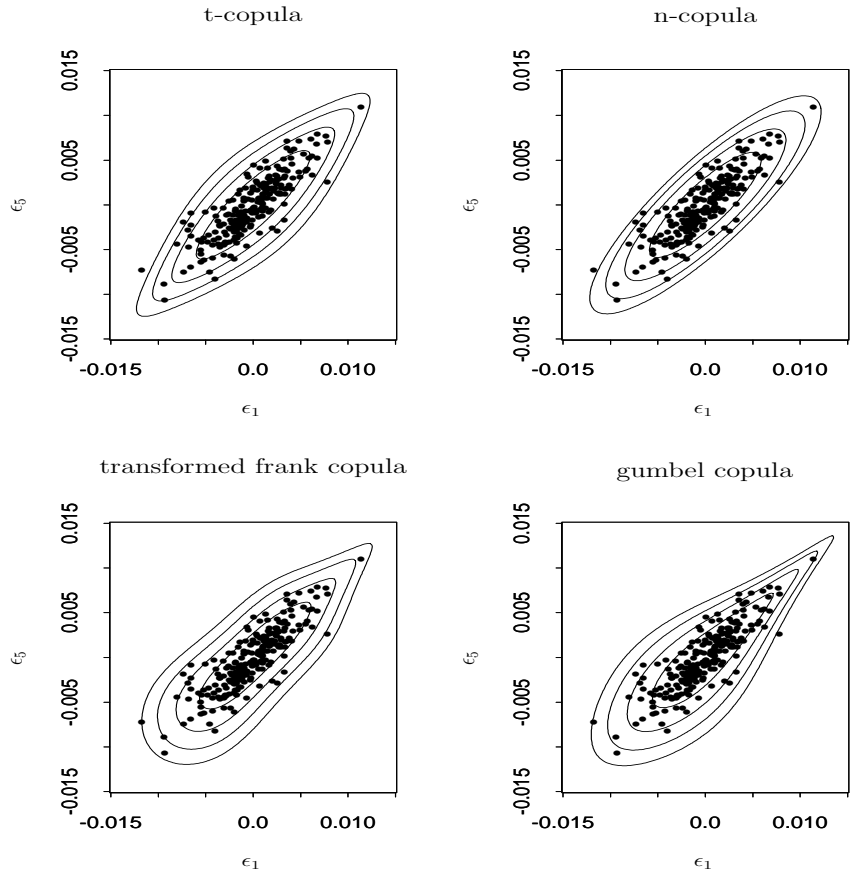
Figure 1: Contourlines of the estimated densities under the different copula models with normal margins. Sample period 1982 to 2001.

t-copula
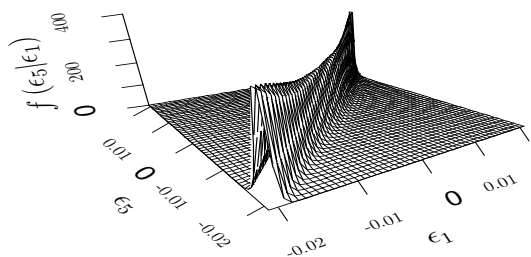
n-copula
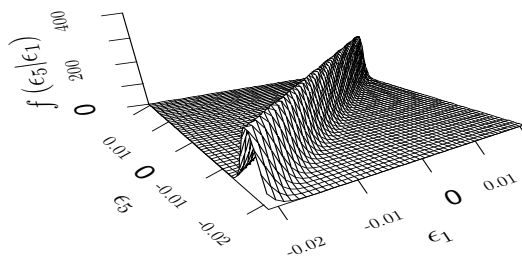
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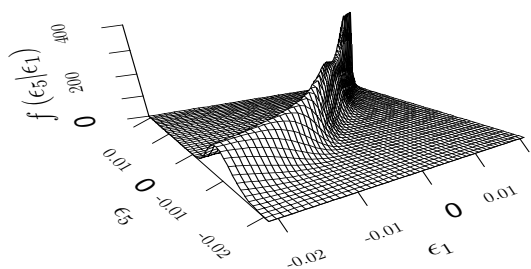
gumbel copula
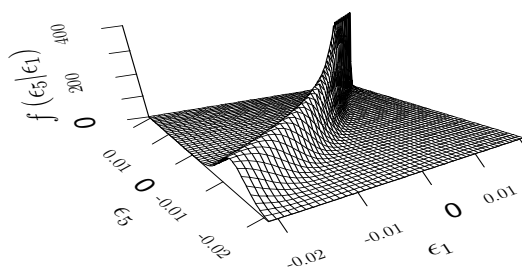
Figure 2: Conditional densities under the different fitted copula functions. Sample period 1982 to 2001.

Relative deviation from the n-copula model
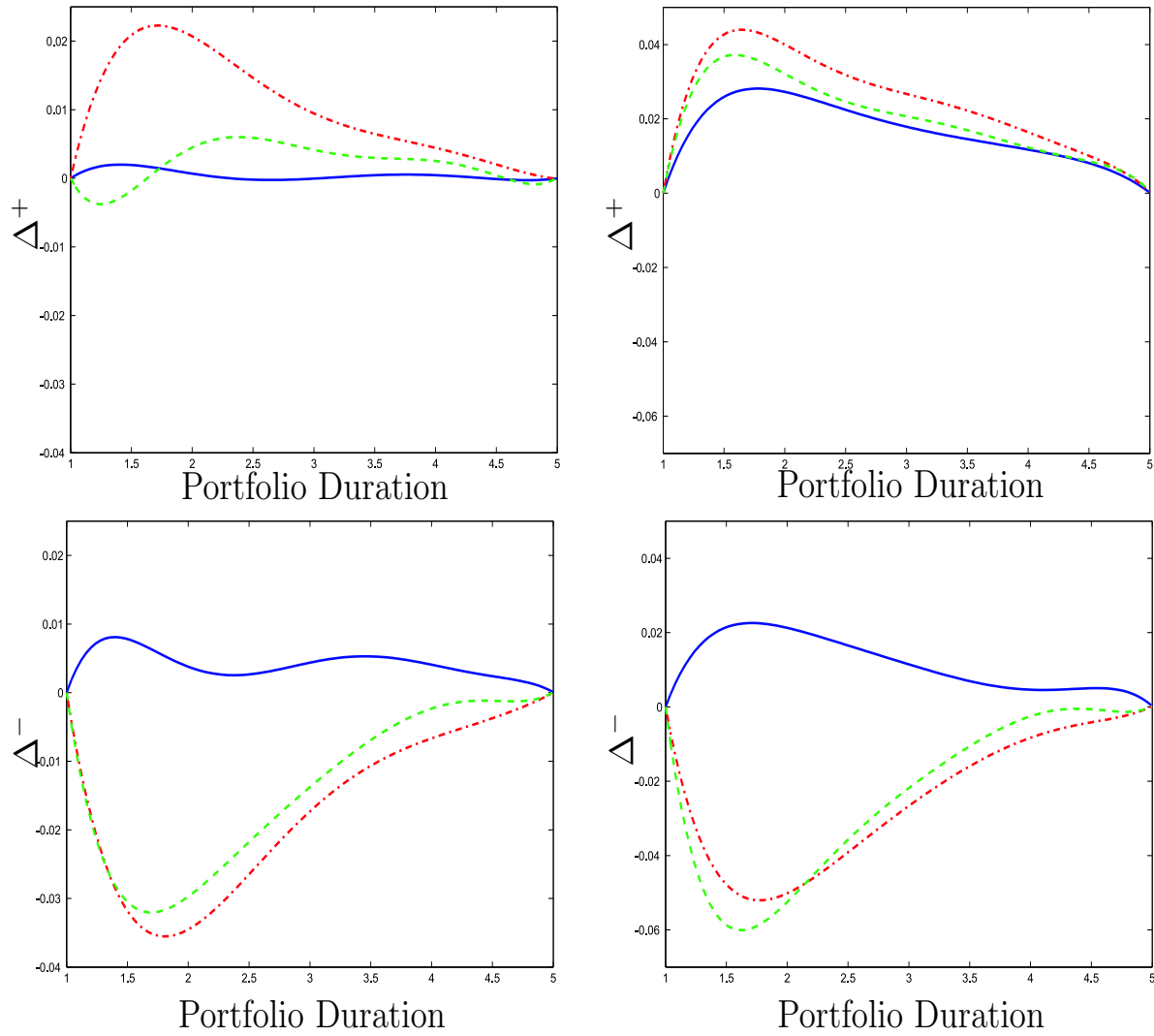
Figure 3: Relative deviations Δ^+ (top) and Δ^- (bottom) of the risk measures from the normal copula model for the $\alpha = 99\%$ (left) and $\alpha = 99.9\%$ (right) quantiles. The solid line belongs to the t-copula, the dashed-dotted line to the Gumbel copula, and the dashed line to the transformed Frank copula. Sample period 1982 to 2001.

Table 1

Summary statistics (sample mean, sample standard deviation, and first order sample autocorrelation) for the monthly U.S. Treasury zero-coupon yield data. Sample period 1982 to 2001 and subsamples 1982 to 1991 and 1992 to 2001.

October 1982 to December 2001			
Maturity	Mean	Std. Dev.	Autocorr.
1 year	0.0638	0.0201	0.9852
2 years	0.0675	0.0206	0.9859
3 years	0.0701	0.0204	0.9854
4 years	0.0721	0.0205	0.9851
5 years	0.0732	0.0205	0.9858

October 1982 to December 1991			
Maturity	Mean	Std. Dev.	Autocorr.
1 year	0.0794	0.0157	0.9641
2 years	0.0837	0.0165	0.9707
3 years	0.0863	0.0165	0.9709
4 years	0.0886	0.0166	0.9696
5 years	0.0897	0.0167	0.9708

January 1992 to December 2001			
Maturity	Mean	Std. Dev.	Autocorr.
1 year	0.0494	0.0109	0.9712
2 years	0.0526	0.0098	0.9581
3 years	0.0550	0.0088	0.9447
4 years	0.0569	0.0082	0.9371
5 years	0.0579	0.0080	0.9387

<div align="center">**Table 2**</div>

Estimation results for the two-factor generalized Vasicek model with monthly observations on 1, 2, 3, 4 and 5 year maturity yields. Sample period 1982 to 2001 and subsamples 1982 to 1991 and 1992 to 2001. Kalman filter recursions are initialized with the values of the stationary mean and variance of the unobserved state variables. Maximization of the log-likelihood function is based on a sequential quadratic programming algorithm. White (1982) heteroskedasticity-consistent standard errors of the parameter estimates given in parenthesis. Lagrange multiplier (LM) test results are reported including p-values and degrees of freedom (df) following Duan and Simonato (1999).

	Kalman Filter Estimates		
Parameters	Oct. 82 to Dec. 01	Oct. 82 to Dec. 91	Jan. 92 to Dec. 01
R_0	0.0589	0.0637	0.0286
	(0.0291)	(0.0254)	(0.0176)
κ_1	0.0691	0.1225	0.1918
	(0.0227)	(0.0285)	(0.0143)
σ_1	0.0203	0.0260	0.2402
	(0.0034)	(0.0048)	(0.0151)
γ_1	-0.1850	-0.0465	-0.4359
	(0.1400)	(0.1696)	(0.1465)
κ_2	0.3719	0.4954	0.2131
	(0.0413)	(0.0610)	(0.0162)
σ_2	0.0188	0.0230	0.2385
	(0.0033)	(0.0047)	(0.0158)
γ_2	1.3358	1.4057	1.5395
	(0.1838)	(0.2509)	(0.2310)
ρ	-0.7807	-0.8199	-0.9991
	(0.0797)	(0.0774)	(0.0001)
$\sigma_{\varepsilon 1}$	0.0014	0.0008	0.0014
	(0.0001)	(0.0002)	(0.0001)
$\sigma_{\varepsilon 2}$	0.0004	0.0006	0.0002
	(0.0001)	(0.0001)	(0.0001)
$\sigma_{\varepsilon 3}$	0.0006	0.0007	0.0003
	(0.0001)	(0.0001)	(0.0001)
$\sigma_{\varepsilon 4}$	0.0006	0.0008	0.0002
	(0.0001)	(0.0001)	(0.0001)
$\sigma_{\varepsilon 5}$	0.0005	0.0006	0.0005
	(0.0001)	(0.0001)	(0.0001)
LM	178.337	68.065	135.972
p-value	0.00000	0.00000	0.00000
df	8	8	8

Table 3

Parametric ML-estimates of the joint distribution function G under the alternative copula models including LR test results. Standard errors and t-values of the parameter estimates given in parenthesis. Sample period 1982 to 2001.

October 1982 to December 2001						
$T = 230$	$\widehat{\sigma}_{1,\cdot}$	$\widehat{\sigma}_{2,\cdot}$	$\widehat{\rho}$	$\widehat{\nu}$	$\widehat{\lambda}_L$	$\widehat{\lambda}_U$
C_t	3.4291E-3	3.4730E-3	0.8556	10.2957	0.3681	0.3681
s.e.	(0.1806E-3)	(0.1701E-3)	(0.0196)	(7.6051)	(0.1823)	(0.1823)
t-value	(18.99)	(20.42)	(43.76)	(1.35)	(2.02)	(2.02)
C_N	3.4517E-3	3.4510E-3	0.8537	-	0	0
s.e.	(0.1845E-3)	(0.1679E-3)	(0.0190)	-	-	-
t-value	(18.71)	(20.55)	(44.97)	-	-	-
	$\widehat{\sigma}_{1,\cdot}$	$\widehat{\sigma}_{2,\cdot}$	$\widehat{\vartheta}$	$\widehat{\delta}$	$\widehat{\lambda}_L$	$\widehat{\lambda}_U$
C_{TF}	3.4152E-3	3.4738E-3	4.1759	1.8101	0	0.5334
s.e.	(0.17785E-3)	(0.1665E-3)	(1.1523)	(0.2463)	-	(0.0764)
t-value	(19.20)	(20.86)	(3.62)	(3.29)	-	(6.98)
C_G	3.5237E-3	3.5433E-3	-	2.8805	0	0.7279
s.e.	(0.18045E-3)	(0.16912E-3)	-	(0.2047)	-	(0.0218)
t-value	(19.53)	(20.95)	-	(9.19)	-	(33.46)
		LR tests	deviance	p-value		
		C_N vs. C_t	2.06	0.0756		
		C_G vs. C_{TF}	13.46	0.0002		

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Table 4

Parametric ML-estimates of the joint distribution function G under the alternative copula models including LR test results for the subsample periods 1982 to 1991 and 1992 to 2001. t-values of the parameter estimates given in parenthesis.

October 1982 to December 1991						
$T = 110$	$\widehat{\sigma}_{1,\cdot}$	$\widehat{\sigma}_{2,\cdot}$	$\widehat{\rho}$	$\widehat{\nu}$	$\widehat{\lambda}_L$	$\widehat{\lambda}_U$
C_t	4.2012E-3	4.0293E-3	0.8871	27.6529	0.2008	0.2008
t-value	(15.33)	(15.01)	(43.13)	(0.36)	(0.31)	(0.31)
C_N	4.2051E-3	4.0202E-3	0.8868	-	0	0
t-value	(15.31)	(15.31)	(43.79)	-	-	-
	$\widehat{\sigma}_{1,\cdot}$	$\widehat{\sigma}_{2,\cdot}$	$\widehat{\vartheta}$	$\widehat{\delta}$	$\widehat{\lambda}_L$	$\widehat{\lambda}_U$
C_{TF}	4.1678E-3	4.0233E-3	1.9274	2.6545	0	0.7016
t-value	(15.91)	(15.50)	(1.06)	(2.61)	-	(8.68)
C_G	4.2200E-3	4.0745E-3	-	3.2772	0	0.7645
t-value	(16.83)	(16.13)	-	(7.24)	-	(30.48)
	LR tests	deviance	p-value			
	C_N vs. C_t	0.11	0.3701			
	C_G vs. C_{TF}	1.17	0.2794			

January 1992 to December 2001						
$T = 119$	$\widehat{\sigma}_{1,\cdot}$	$\widehat{\sigma}_{2,\cdot}$	$\widehat{\rho}$	$\widehat{\nu}$	$\widehat{\lambda}_L$	$\widehat{\lambda}_U$
C_t	2.5580E-3	2.9075E-3	0.8148	6.2034	0.4188	0.4188
t-value	(13.60)	(14.68)	(23.83)	(1.95)	(3.22)	(3.22)
C_N	2.5634E-3	2.7780E-3	0.7886	-	0	0
t-value	(13.75)	(16.48)	(21.52)	-	-	-
	$\widehat{\sigma}_{1,\cdot}$	$\widehat{\sigma}_{2,\cdot}$	$\widehat{\vartheta}$	$\widehat{\delta}$	$\widehat{\lambda}_L$	$\widehat{\lambda}_U$
C_{TF}	2.5364E-3	2.8771E-3	5.8523	1.3362	0	0.3201
t-value	(13.73)	(16.31)	(2.11)	(0.91)	-	(1.33)
C_G	2.5802E-3	2.9629E-3	-	2.4207	0	0.6684
t-value	(13.62)	(15.09)	-	(6.02)	-	(17.98)
	LR tests	deviance	p-value			
	C_N vs. C_t	14.16	0.0001			
	C_G vs. C_{TF}	13.12	0.0003			

Table 5

Parametric ML-estimates of the joint distribution function G under the alternative copula models including LR test results for the unrestricted and smoothed affine systems. t-values of the parameter estimates given in parenthesis. Sample periods 1982 to 2001.

October 1982 to December 2001 (unrestricted system)						
$T = 230$	$\widehat{\sigma}_{1,\cdot}$	$\widehat{\sigma}_{2,\cdot}$	$\widehat{\rho}$	$\widehat{\nu}$	$\widehat{\lambda}_L$	$\widehat{\lambda}_U$
C_t	3.4182E-3	3.4725E-3	0.8554	8.8836	0.4009	0.4009
t-value	(18.86)	(20.88)	(44.49)	(2.24)	(3.54)	(3.54)
C_N	3.4405E-3	3.4491E-3	0.8531	-	0	0
t-value	(18.52)	(20.54)	(44.62)	-	-	-
	$\widehat{\sigma}_{1,\cdot}$	$\widehat{\sigma}_{2,\cdot}$	$\widehat{\vartheta}$	$\widehat{\delta}$	$\widehat{\lambda}_L$	$\widehat{\lambda}_U$
C_{TF}	3.4144E-3	3.4613E-3	3.7458	1.8957	0	0.5586
t-value	(18.72)	(20.93)	(3.24)	(3.37)	-	(7.56)
C_G	3.5304E-3	3.5307E-3	-	2.8646	0	0.7262
t-value	(19.01)	(20.88)	-	(9.21)	-	(33.33)
		LR tests	deviance	p-value		
		C_N vs. C_t	4.09	0.0216		
		C_G vs. C_{TF}	15.21	0.0000		

October 1982 to December 2001 (smoothed affine system)						
$T = 230$	$\widehat{\sigma}_{1,\cdot}$	$\widehat{\sigma}_{2,\cdot}$	$\widehat{\rho}$	$\widehat{\nu}$	$\widehat{\lambda}_L$	$\widehat{\lambda}_U$
C_t	3.4608E-3	3.4019E-3	0.9328	4.6824	0.6731	0.6731
t-value	(22.09)	(22.33)	(111.24)	(5.91)	(21.29)	(21.29)
C_N	3.4763E-3	3.3708E-3	0.9300	-	0	0
t-value	(20.85)	(20.81)	(95.89)	-	-	-
	$\widehat{\sigma}_{1,\cdot}$	$\widehat{\sigma}_{2,\cdot}$	$\widehat{\vartheta}$	$\widehat{\delta}$	$\widehat{\lambda}_L$	$\widehat{\lambda}_U$
C_{TF}	3.4424E-3	3.4117E-3	4.9431	2.4687	0	0.6759
t-value	(20.93)	(20.70)	(3.41)	(3.48)	-	(10.64)
C_G	3.5219E-3	3.5342E-3	-	4.2494	0	0.8228
t-value	(20.92)	(21.04)	-	(10.66)	-	(59.75)
		LR tests	deviance	p-value		
		C_N vs. C_t	4.33	0.0187		
		C_G vs. C_{TF}	15.02	0.0001		

Table 6

Goodness-of-fit statistics for the ML-estimates of the joint distribution function G under the alternative copula models. $^+$indicates best model fit for a given statistic, $^-$indicates worst model fit for a given statistic. Sample period 1982 to 2001 and subsamples 1982 to 1991 and 1992 to 2001.

October 1982 to December 2001				
$T = 230$	C_t	C_N	C_{TF}	C_G
$\chi^2_{23,24}$ (p-value)	$[0.06, 0.21]^-$	$[0.07, 0.19]$	$[0.41, 0.51]^+$	$[0.07, 0.19]$
AIC	-4204.30	-4204.24	-4215.34^+	-4203.88^-
BIC	-4188.54^-	-4191.93	-4199.58^+	-4191.57
EN	-9.15	-9.15	-9.19^+	-9.14^-
AD	21.61^-	0.69	0.52^+	0.63
$PD_{p=0.05}$	$22.58\%^-$	20.26%	$6.19\%^+$	9.67%
$\overline{PD}_{p=0.05}$	-4.08%	-7.21%	$-3.67\%^+$	$21.76\%^-$

October 1982 to December 1991				
$T = 110$	C_t	C_N	C_{TF}	C_G
$\chi^2_{12,13}$ (p-value)	$[0.76, 0.79]$	$[0.84, 0.85]^+$	$[0.72, 0.75]$	$[0.69, 0.72]^-$
AIC	-1954.91^-	-1956.80	-1960.44	-1961.27^+
BIC	-1942.11^-	-1946.70	-1947.64	-1951.17^+
EN	-8.92	-8.91^-	-8.96^+	-8.94
AD	14.6768^-	0.6751	0.6134^+	0.6489
$PD_{p=0.1}$	$7.25\%^-$	6.36%	$-0.91\%^+$	-1.86%
$\overline{PD}_{p=0.1}$	0.57%	$0.40\%^+$	7.66%	$14.29\%^-$

January 1992 to December 2001				
$T = 119$	C_t	C_N	C_{TF}	C_G
$\chi^2_{12,13}$ (p-value)	$[0.58, 0.65]$	$[0.42, 0.52]$	$[0.67, 0.71]^+$	$[0.40, 0.50]^-$
AIC	-2267.39^+	-2255.23	-2265.22	-2254.10^-
BIC	-2244.28	-2244.89	-2252.10^+	-2243.76^-
EN	-9.49	-9.51	-9.56^+	-9.47^-
AD	6.0714^-	0.2756^+	0.3732	0.2796
$PD_{p=0.1}$	$0.72\%^+$	-9.81%	-7.35%	$-12.81\%^-$
$\overline{PD}_{p=0.1}$	4.35%	-5.68%	$0.41\%^+$	$16.22\%^-$

Table 7

Goodness-of-fit statistics for the ML-estimates of the joint distribution function G under the alternative copula models. $^{+}$indicates best model fit for a given statistic, $^{-}$indicates worst model fit for a given statistic. Sample period 1982 to 2001. Unrestricted system and smoothed affine system.

October 1982 to December 2001 - unrestricted system				
$T = 230$	C_t	C_N	C_{TF}	C_G
$\chi^2_{23,24}$ (p-value)	[0.01, 0.12]	[0.01, 0.08]$^{-}$	[0.10, 0.25]$^{+}$	[0.02, 0.12]
AIC	-4374.37	-4372.28	-4384.81^{+}	-4371.60^{-}
BIC	-4360.62^{-}	-4361.97	-4371.06^{+}	-4361.29
EN	-9.59^{+}	-9.54	-9.57	-9.48^{-}
AD	31.93^{-}	0.55	0.45^{+}	0.55
$PD_{p=0.05}$	21.08%$^{-}$	17.69%	3.79%$^{+}$	7.47%
$\overline{PD}_{p=0.05}$	1.81%$^{+}$	-2.38%	3.16%	26.79%$^{-}$

October 1982 to December 2001 - smoothed affine system				
$T = 230$	C_t	C_N	C_{TF}	C_G
$\chi^2_{23,24}$ (p-value)	[0.60, 0.65]	[0.51, 0.57]$^{-}$	[0.86, 0.86]$^{+}$	[0.77, 0.78]
AIC	-4374.61	-4372.28	-4384.81^{+}	-4371.79^{-}
BIC	-4359.85^{-}	-4361.97	-4371.06^{+}	-4362.48
EN	-9.50^{-}	-9.52	-9.54^{+}	-9.51
AD	30.23^{-}	0.57	0.45^{+}	0.53
$PD_{p=0.05}$	11.19%$^{-}$	4.74%	-6.54%	0.56%$^{+}$
$\overline{PD}_{p=0.05}$	-15.03%$^{-}$	-9.03%$^{+}$	-11.46%	9.68%