Using Simulation for Option Pricing

John Charnes
U. Kansas School of Business
Presentation to Winter Simulation Conference
Orlando, FL
12 December 2000
Introduction

- Increased complexity of numerical computation in financial theory and practice has put demands on computational speed and efficiency
- Monte Carlo is useful for valuation of securities, estimation of their sensitivities, risk analysis, and stress testing of portfolios
Session Objectives

- Demonstrate how simulation is used for pricing derivatives
- Describe how variance reduction techniques increase precision of estimates without increasing number of runs
- Provide examples of various financial options for use by educators and others interested in applying simulation to derivative pricing with spreadsheet models
Agenda

- Overview
- Variance reduction and efficiency improvement
- Low-discrepancy sequences
- Conclusion
Overview

- Monte Carlo used for
  - Stochastic volatility applications
  - Valuation of mortgage-backed securities
  - Valuation of path-dependent options
  - Portfolio optimization
  - Interest-rate derivative claims

- Increased use by practitioners has sparked methodological developments in variance reduction and low-discrepancy sequences
Vocabulary

- **Risk-neutral pricing**
  - With no-arbitrage assumption, price of derivative security can be expressed as the expected value of its payouts discounted at risk-free rate of interest

- **Equivalent martingale measure**
  - Expectation is taken with respect to a transformation of the original probability measure
Risk-Neutral Pricing

- If option is European,
  \[ C_E = E[e^{-rT}(S_T - K)^+] \]
  - Can be found by Black-Scholes Formula

- For American option,
  \[ C_A = \max_{\tau} E[e^{-r\tau}(S_\tau - K)^+] \]
  over all stopping times \( \tau \leq T \)
  - Cannot be found by Black-Scholes
Simulating European Options

- Purpose
  - Even though simulation is not necessary to determine fair price of European options, it is used with European options to test algorithms and variance reduction techniques.
How To Price European Options with Simulation

- Simulate future stock price using risk-free rate of growth and assumed level of volatility
- Evaluate discounted (at risk-free rate) cash flow for each simulated price
- Average discounted cash flows over iterations of simulation
Lognormal Model for Future Price

\[
S_{t+\Delta t}^{(i)} = S_t \exp\left[\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}Z^{(i)}\right]
\]

for replication \( i = 1, \ldots, n \); where

- \( S_t \) is stock price at time \( t \)
- \( r \) is the risk-free rate of interest
- \( \sigma \) is the volatility
- \( \Delta t \) is the time step
- \( Z^{(i)} \sim N(0,1) \)
Example

- EuroCall.xls
- How many iterations (\(n\)) must be run to achieve a specified precision?

\[
\bar{x} \pm 2 \sqrt{\frac{\sigma^2}{n}}
\]

- Precision is increased with larger \(n\) or smaller \(\sigma^2\)
Variance Reduction Techniques

- Antithetic Variates
- Control Variates
- Moment Matching
- Latin Hypercube Sampling
- Importance Sampling
- Conditional Monte Carlo
- Quasi-Monte Carlo
Antithetic Variates

- Take average of two separate estimators that are designed to have negative correlation
- EuroCallAV.xls
Control Variates

- Replace the evaluation of an unknown expectation with the evaluation of the difference between the unknown quantity and a related quantity whose expectation is known.

- AsianCallCV.xls
  - Known expectation is price of an Asian option that pays off on geometric average.
  - Unknown expectation is price of an Asian option that pays off on arithmetic average.
Latin Hypercube Sampling

- Select $z^{(i)}$'s randomly from each of $k$ intervals having area under curve $= 1/k$
- EuroCallLHS.xls
Moment-Matching

- **EuroCallMM.xls**
  - Generate sample terminal prices, then transform so that sample moments equal population moments
- **Boyle et al. (1997) show that** whenever a population moment is known, it’s better to use it as a control variate than for Moment Matching
Low-discrepancy Sequences

- Discrepancy measures the extent to which points are evenly dispersed throughout a region—the more evenly dispersed the lower the discrepancy.
- Low-discrepancy sequences are also known as quasi-random sequences even though they are not at all random.
Quasi-random Sequence

- For any integer \( n \), and any prime number \( r \geq 2 \):
  - Expand \( n \) in terms of \( m \) places in base \( r \)
    \[
    n = \sum_{j=0}^{m} a_j(n) r^j
    \]
  - Quasi-random number is “reflection about the decimal point”
    \[
    \phi_r = \sum_{j=0}^{m} a_j(n) r^{-j-1}
    \]
Example (base 3):

<table>
<thead>
<tr>
<th>n</th>
<th>Base 3</th>
<th>$\phi_3(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>01</td>
<td>1/3</td>
</tr>
<tr>
<td>2</td>
<td>02</td>
<td>2/3</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>1/9</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
<td>4/9</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>7/9</td>
</tr>
</tbody>
</table>
Example (base 3):

<table>
<thead>
<tr>
<th></th>
<th>Base 3</th>
<th>$\phi_3(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>20</td>
<td>2/9</td>
</tr>
<tr>
<td>7</td>
<td>21</td>
<td>5/9</td>
</tr>
<tr>
<td>8</td>
<td>22</td>
<td>8/9</td>
</tr>
<tr>
<td>9</td>
<td>100</td>
<td>1/27</td>
</tr>
<tr>
<td>10</td>
<td>101</td>
<td>4/27</td>
</tr>
</tbody>
</table>

- Added points “know” how to fill the gaps evenly
s-dimensional QR Sequence

- Let r be smallest prime number \( \geq s \)
  - Represent \( n \) in base \( r \)
    \[
    n = \sum_{j=0}^{m} a_j^1(n) r^j
    \]
  - Find remaining elements recursively
    \[
    a_j^n(n) = \sum_{i \leq j} \binom{i}{j} a_i^{k-1}(n) r \mod r
    \]
Quasi-Monte Carlo

- Gives spectacular reductions in MSE
- EuroCallQMC.xls
American Put Option

- A stock has price today = $S_0$
- A put option is available for purchase that gives owner the right to sell stock for strike price, $K$, at any time $t$, $0 \leq t \leq T$
- What is the fair value, $P$, of the put option?
American Put Option

- Early exercise feature makes valuation difficult

\[ P = \max_{\tau} \mathbb{E}[e^{-r\tau} (K - S_\tau)^+] \]

over all stopping times \( \tau \leq T \)

- In practice, find value of Bermudan put option, which can be exercised only at a finite number of opportunities, \( k \), before expiration
Valuing Bermudan Put Options

- Analytical solution given by Geske and Johnson, JF 1984, for small $k$
- Simulation approach given by Broadie and Glasserman, JEDC 1997, for small $k$, and
- See also Fu, et al. (1999)
- Forward Monte Carlo method (Charnes and Shenoy 2000)
- Using OptQuest, package for stochastic optimization using tabu search (Glover 1997)
Free-Boundary Problem

- For each exercise opportunity, must estimate price below which put option should be exercised and above which put option should be held
- BermuPutOptAV.xls
- Uses tabu search to identify an optimal policy, then a final set of iterations to estimate value under optimal policy
Conclusion

- Interest in Monte Carlo methods for option pricing increasing because of its flexibility in handling complex derivatives
- As workstations get faster, will be able to value increasingly complex financial instruments in smaller amounts of computer time