CURRENCY OPTION PRICING II

- Calibrating the Binomial Tree to Volatility
- Black-Scholes Model for Currency Options
  - Properties of the BS Model
- Option Sensitivity Analysis
  - Delta
  - Gamma
  - Vega
  - Theta
  - Rho
Calibrating the Binomial Tree

Instead of $u$ and $d$, you will usually obtain the volatility, $\sigma$, either as

- **Historical volatility**
  - Compute the standard deviation of daily log return ($\ln(S_t / S_{t-1})$)
  - Multiply the daily volatility by $\sqrt{252}$ to get annual volatility

- **Implied volatility**
  - In reality, both types of volatility are available from, e.g., Reuters.
  - In fact, OTC options are quoted by implied volatility.

Euro options quotes, Reuters, 11/27/2003

<table>
<thead>
<tr>
<th>Time</th>
<th>Bid</th>
<th>Ask</th>
<th>Bank</th>
<th>Location</th>
<th>Year</th>
</tr>
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<tr>
<td>3M</td>
<td>10.05</td>
<td>10.90</td>
<td>RBS</td>
<td>LON</td>
<td>18.15</td>
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<td>TOKYO MITSUB</td>
<td>TOK</td>
<td>TMOT</td>
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<td>TOK</td>
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<td>10.80</td>
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<td>LON</td>
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<td>10.85</td>
<td>11.05</td>
<td>RBS</td>
<td>LON</td>
<td>18.05</td>
</tr>
</tbody>
</table>

- To construct a binomial tree that is consistent with a given volatility, set

$$u = e^{\sigma \sqrt{dt}}, \quad d = e^{-\sigma \sqrt{dt}},$$

where $dt = \frac{\tau}{n}$ is the length of a time step (a subtree) and $n$ is the number of time steps (subtrees) until maturity.
**Example.** Let us construct a one-period tree that is consistent with a 10% volatility. Using this tree, we will price call and put options.

- Spot $S = 1.15$/€
- Strike $K = 1.15$/€
- Domestic interest rate $i^S = 1.2\%$ (continuously compounded)
- Foreign interest rate $i^€ = 2.2\%$ (continuously compounded)
- Volatility $\sigma = 10\%$ (annualized)
- Time to maturity = 6 months ($\tau = 0.5$)

$u = \exp(0.1 \times \sqrt{0.5}) = 1.0773$

$d = \exp(-0.1 \times \sqrt{0.5}) = 0.93173$

$q = [\exp((0.012 - 0.022) \times 0.5) - d]/(u - d) = 0.44709$

Risk neutral pricing gives

\[
C = [0.08426 \times q + 0 \times (1 - q)] \times \exp(-0.012 \times 0.5) = 0.03745
\]

\[
P = [0 \times q + 0.07851 \times (1 - q)] \times \exp(-0.012 \times 0.5) = 0.04315
\]
As the number of time steps increases \((n \to \infty \text{ and } dt \to 0)\), the binomial model price converges to the Black-Scholes price.

<table>
<thead>
<tr>
<th>Time step (n)</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>50</th>
<th>100</th>
<th>500</th>
<th>BS</th>
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<tr>
<td>Call</td>
<td>0.0374</td>
<td>0.0310</td>
<td>0.0286</td>
<td>0.0292</td>
<td>0.0293</td>
<td>0.0294</td>
<td>0.0294</td>
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<tr>
<td>Put</td>
<td>0.0431</td>
<td>0.0367</td>
<td>0.0343</td>
<td>0.0349</td>
<td>0.0350</td>
<td>0.0351</td>
<td>0.0351</td>
</tr>
</tbody>
</table>

Graphically:

![Convergence of the Binomial to BS Model](image_url)
Black-Scholes Model for Currency Options

To price currency options, you can use the Black-Scholes formula on a dividend-paying stock with the \textit{dividend yield} replaced by the \textit{foreign interest rate}.\footnote{This is known as the Garman-Kohlhagen model}

\begin{align*}
C &= S_0 e^{-i_d \tau} N(d_1) - K e^{-i_f \tau} N(d_2), \\
P &= K e^{-i_f \tau} N(-d_2) - S_0 e^{-i_d \tau} N(-d_1)
\end{align*}

where

\begin{align*}
d_1 &= \frac{\ln(S/K) + (i_d - i_f + \frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{\tau}}, \\
d_2 &= \frac{\ln(S/K) + (i_d - i_f - \frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{\tau}} \\
&= d_1 - \sigma \sqrt{\tau}
\end{align*}
Note that, in the FX context, you can write the formula in terms of the forward rate so that the foreign interest rate (or even the spot rate!) does not appear.

Since $F = S e^{(r^d - r^f)τ}$,

$$C = e^{-r^fτ}[FN(d_1) - KN(d_2)], \quad P = e^{-r^fτ}[KN(-d_2) - FN(-d_1)]$$

(3)

where

$$d_1 \equiv \frac{\ln(F / K) + \frac{1}{2} \sigma^2 τ}{\sigma \sqrt{τ}}, \quad d_2 \equiv \frac{\ln(F / K) - \frac{1}{2} \sigma^2 τ}{\sigma \sqrt{τ}}$$

$$= d_1 - \sigma \sqrt{τ}$$

- Discounting by the risk-free rate in Equation (3) indicates that the terms in the square brackets are certainty equivalent of the option payoff at maturity.
- Note: you do have to use the forward rate that corresponds to the maturity of the option.

Example.

- Spot $S = 1.15$/€
- Strike $K = 1.15$/€
- Domestic interest rate $i^d = 1.2\%$ (continuously compounded)
- Foreign interest rate $i^e = 2.2\%$ (continuously compounded)

(Or you might observe the forward rate, $F = 1.1443$/€. Then use (3)-(4))

- Volatility $\sigma = 10\%$
- Time to maturity = 6 months

$$d_1 = \frac{\ln(1.15/1.15) + (.012 - .022 + .12/2)\times.5}{(.1\times.5)\times.5} = -.035355$$

$$d_2 = d_1 - .1\times.5 = -.10607$$

$$N(d_1) = .48590, \quad N(-d_1) = 1 - N(d_1) = .51410$$

$$N(d_2) = .44776, \quad N(-d_2) = 1 - N(d_2) = .54224$$

$$C = 1.15\times e^{-22\times.5} \times .48590 - 1.15\times e^{-12\times.5} \times .44776 = .02939$$

$$P = 1.15\times e^{-12\times.5} \times .54224 - 1.15\times e^{-22\times.5} \times .51410 = .03509$$
Properties of the Black-Scholes Model for Currency Options

These are also the properties of the BS model on a dividend-paying stock.

1. The B-S model assumes the future spot rate is distributed lognormally. Without this strong assumption we can still put a lower bound on a European call option:

\[ C \geq \text{Max}(S \exp(-i^f \tau) - K \exp(-i^d \tau), 0) \]

- To see this, let us first derive the following important result:

**Result 1**

| The present value of receiving \( S_T \) at maturity is \( S_T \exp(-i^f \tau) \). |

- Note: this is **NOT** \( S_t \). The value of foreign interest is subtracted.

This can be seen by considering the following investment strategy (take currency \( d = \$ \), currency \( f = \€ \)):

- Now: invest \( \$ S_t \exp(-i^f \tau) = \€ \exp(-i^f \tau) \) in a euro deposit. Reinvest the interest continuously in the deposit itself.

- At maturity: you will receive \( \€ 1 (=\exp(-i^f \tau) \times \exp(i^f \tau)) \) or equivalently \( \$ S_T \).

- The payoff of a call option at maturity is the larger of \( S_T - K \) or 0.

- The PV of receiving \( S_T - K \) **at maturity** is, from the above result,

\[ S_T \exp(-i^f \tau) - K \exp(-i^d \tau) \]

- Since the value of a call option is never negative, we have the above inequality.

- Graphically:
2. The prices of **forward ATM call** and **put** options are the same.
   - Graphically, recall that we can create a **synthetic forward** by a call and a put.
     - Since the forward costs nothing, the price of the call and the put must balance.

   Mathematically, since we always have
   \[
   N(-d_2) - N(-d_1) = 1 - N(d_2) - [1 - N(d_1)] = N(d_1) - N(d_2),
   \]
   if \( K = F \) in (3),
\[ C = e^{-\rho \tau} F[N(d_1) - N(d_2)], \quad P = e^{-\rho \tau} F[N(-d_2) - N(-d_1)] \]
\[ = e^{-\rho \tau} F[N(d_1) - N(d_2)] = C. \]

* Note: Equation (4) also becomes very simple:
\[ d_1 = \sigma \sqrt{\tau} / 2, \quad d_2 = -\sigma \sqrt{\tau} / 2 = -d_1. \]
Thus,
\[ N(d_2) = 1 - N(-d_2) = 1 - N(d_1) \]
and therefore we can further write
\[ C = P = e^{-\rho \tau} F[2N(d_1) - 1]. \]
3. **Digital (Binary) Options.**

- **Asset-or-nothing** (AON) and **cash-or-nothing** (CON) options are not really exotic. They are the **basic building blocks** of the Black-Scholes Model. By definition,

\[
C = AON(S_T > K) - K \cdot CON(S_T > K)
\]

\[
P = K \cdot CON(S_T < K) - AON(S_T < K)
\]

Compare with (1). We obtain:

\[
AON(S_T > K) = S e^{-rT} N(d_1), \quad CON(S_T > K) = e^{-rT} N(d_2)
\]

\[
AON(S_T < K) = S e^{-rT} N(-d_1), \quad CON(S_T < K) = e^{-rT} N(-d_2)
\]

From the expressions for CON, we know that the probability of

- the **call** ending up **in the money** \((S_T > K)\) is \(N(d_2)\),
- the **put** ending up **in the money** \((S_T < K)\) is \(N(-d_2)\).

**Risk-neutral probabilities of the call and the put ending up ITM**
Option Sensitivity Analysis

Delta

The delta of an option (or a portfolio), $\Delta$, is the rate of change in the price of the option (or portfolio) with respect to the spot rate.

- Delta of a **European call**:  
  $$\Delta_{\text{Call}} = \frac{\partial C}{\partial S} = e^{-i\tau} N(d_1)$$

  - Recall that $0 < \Delta_{\text{Call}} < 1$. In the B-S world, a tighter bound obtains:  
    $$0 < \Delta_{\text{Call}} < \exp(-i\tau) < 1.$$

- Delta of a **European put**:  
  $$\Delta_{\text{Put}} = \frac{\partial P}{\partial S} = -e^{-i\tau} N(-d_1)$$

  $$-1 < -\exp(-i\tau) < \Delta_{\text{Put}} < 0.$$

**Delta of the European call option in the Example**

As the figure shows,

- $\Delta_{\text{Call}} \to \exp(-i\tau)$ (closer to 1) as $S_t \to \infty$ (ITM).
- $\Delta_{\text{Call}} \to 0$ as $S_t \to 0$ (OTM).

Q1. What is the limit of the delta of a put as $S_t \to \infty$ or $S_t \to 0$?
Q2. In the above example, recall that \( S = 1.15 \) and \( F = 1.1443 \).

- Why does the delta of the 1.25 call converge to 0?
- Why does the delta of the 1.05 call become closer to 1?

- The delta of a spot contract is 1 (confirm this).
- The delta of a forward contract is \( \exp(-i_f \tau) \).

Q3. Show this by first writing down the value of a long forward contract (this is different from the formula for the forward rate) and then differentiating it with respect to the spot rate.

- The delta of a futures contract is \( \exp((d - d^f) \tau) \). This is slightly different from the forward contract because of the mark-to-market.
- The mark-to-market enables you to realize the gain or loss caused by the change in the spot rate immediately (daily).
- The delta of a portfolio is the sum of the deltas of its component assets.
Example, continued. The delta of the above call option is
\[ \exp(-i \tau) N(d_1) = \exp(-2.2\% \times 0.5) \times 0.48590 = 0.4806. \]
The delta of the put option is
\[ -\exp(-i \tau) N(-d_1) = -\exp(-2.2\% \times 0.5) \times 0.51410 = -0.5085. \]

Q4. A bank has sold the above put option on €1 million. How can the bank make its position delta neutral using the CME futures contract? The size of the CME euro futures contract is €125,000. What cares should be taken about this delta hedge?

Gamma

- The gamma of an option/portfolio is the rate of change of the option/portfolio’s delta with respect to the spot rate.
- It is the second partial derivative of the option/portfolio price with respect to the spot rate.
- It measures the curvature of the relation between the option/portfolio price and the spot rate.
- The gamma of a European call option and a put option with the same strike price turns out to be the same:
  \[ \Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial^2 P}{\partial S^2} = \frac{e^{-i \tau} n(d_1)}{S \sigma \sqrt{\tau}} \]
  where \( n(\cdot) \) is the standard normal density function.

Q5. Show this equivalence (not the formula) by the put-call parity.

- The gammas of a spot, forward, and futures contract are all zero.
- You should be able to derive this.
Gamma of the European call in the example

Graphically, this is the slope of the graph of the delta. Confirm this.

- **Delta neutrality** provides protection against relatively small spot rate movements.
- **Delta-Gamma neutrality** provides protection against larger spot price movements.
  - Traders typically make their position delta-neutral at least once a day.
  - Gamma and vega (see below) are more difficult to zero away, because they cannot be altered by trading the spot, forward, or futures contract.
  - Fortunately, as time elapses, options tend to become deep OTM or ITM. Such options have negligible gamma and vegas.
Vega

The vega of an option/portfolio, \( \nu \), is the rate of change of the option/portfolio value with respect to the volatility of the spot rate.

- The vega of a regular European or American option is always positive.
  - Intuitively, the insurance (time) value of an option increases with volatility.
- The vegas of a European call and a put are the same.

Q6. Show this using put-call parity.

**Vega of the European call in the example**

![Vega vs Spot Rate Graph](image_url)
Theta

The theta of an option/portfolio, $\Theta$, is the rate of change of the option/portfolio value with respect to the passage of time.

- Theta is usually negative for an option, because the passage of time decreases the time value.
- For an ITM call option on a currency with a relatively high interest rate, theta can be positive, because the passage of time shortens the time until the receipt of the foreign interest if exercised in the money (see the picture below).
- Theta is not the same type of hedge measure as others Greeks, because there is no uncertainty about the passage of time.

Theta of the European call in the example

![Theta vs Spot Rate graph](chart.png)
Rho

The rho of an option/portfolio is the rate of change of the option/portfolio value with respect to the interest rate.

- For currency options, there are two rhos, one corresponding to the change in the domestic interest rate and the other corresponding to the foreign interest rate.
- The rho of a European call option with respect to the domestic interest rate is positive. It is negative for a put.
- The rho of a European call option with respect to the foreign interest rate is negative. It is positive for a put.

Q7. Explain the above two points.

Rho of the European call w.r.t. the domestic interest rate in the example

![Rho vs Spot Rate Graph](image-url)
Suggested solutions to questions

Q1. \( \Delta_{\text{Put}} \to 0 \) as \( S_t \to \infty \) (OTM).
\( \Delta_{\text{Put}} \to -\exp(-i^f \tau) \) (closer to -1) as \( S_t \to 0 \) (ITM).

Q2. The delta of the 1.25 call converges to 0 because, as \( \tau \to 0 \), it becomes progressively sure that the option will end up OTM. On the other hand, the delta of the 1.05 call becomes closer to 1 because it becomes very likely that the option will end up ITM.

Q3. The value of a forward contract is
\[
f = S \exp(-i^f \tau) - K \exp(-i^d \tau)
\]
Thus, the delta of a forward contract is \( \frac{\partial f}{\partial S} = \exp(-i^f \tau) \).

Q4. The delta of the futures contract is
\[
\exp((1.2\% - 2.2\%) \times 0.5) = 0.9950.
\]
Let \( x \) be the amount of futures contracts that the bank should hold long. We set
\[
-0.5085 \times (-1 \text{ million}) + 0.9950 \times x = 0.
\]
\[
x = -0.5111 \text{ million} \quad \Rightarrow \quad x / 125,000 = 4.088
\]
Thus, the bank should sell four CME euro futures contracts. As in the Dozier case, this is not a perfect hedge.

- The bank must rebalance dynamically to remain delta-hedged.
- There is a size mismatch.

Q5. By the put-call parity, we have
\[
C - P = S \exp(-i^f \tau) - K \exp(-i^d \tau). \quad (*)
\]
Differentiating with respect to \( S \) gives
\[
\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = e^{-i^f \tau}.
\]
That is, the difference between the deltas of the call and the put equals the foreign discount factor. Further differentiation yields
\[
\frac{\partial^2 C}{\partial S^2} - \frac{\partial^2 P}{\partial S^2} = 0.
\]
This shows the equivalence of the two deltas.
Q6. The put-call parity relation (*) above does not involve $\sigma$. Thus, differentiating with respect to $\sigma$ gives
$$\frac{\partial C}{\partial \sigma} - \frac{\partial P}{\partial \sigma} = 0.$$  

Q7. Equations (3) and (4) can be considered a B-S model “on a forward contract.” A higher domestic interest rate or a lower foreign interest rate will increase the forward rate and therefore makes the call option more ITM, and the put more OTM.