Bayesian Optimal Portfolio Selection: the Black-Litterman Approach

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1 Review and Model Assumptions

Mean-Variance optimal portfolios often tends to behave badly because of their sensitivity to movements in the variance-correlation matrix. Variance and correlation forecasting is notoriously difficult and the high sensitivity of the optimal solution to such inputs often results in extreme (corner) solutions. One possible interpretation of this phenomenon is the error-maximizing tendency of the optimal solution in that assets with positive pricing errors are significantly over-weighted versus those with negative errors, see Michaud (1989, 1998). Further, if there exists an asset with very low volatility relative to other assets, a risk-minimizing procedure will tend to rely too much on that assets rather than diversifying across a wide range of holdings. The Black-Litterman (1992) model can help to construct stable mean-variance efficient portfolios; the model was developed in Goldman Sachs in the early 90s and provides a framework for combining subjective investors views with market (equilibrium) views. It then construct optimal portfolio weights based on a volatility/correlation matrix as in mean-variance analysis.

1.1 Bayesian Updating

In the Black-Litterman (1992) context we shall consider a framework to assess the joint likelihood of investors subjective views (or prior beliefs) and the empirical data (or model-based estimates). Therefore we can imagine that CAPM-implied equilibrium returns (based on data) can be synthesized with currently held opinions by the investment managers to form new opinions. This is a natural way of thinking since it is often the case that practitioners exhibit the most strikingly different views on expected returns compared to the market consensus.

Let us consider two possible events:

$$\begin{aligned} A &= \text{expected return} \\ B &= \text{equilibrium return} \end{aligned}$$

Using Bayes Law we can decompose the joint likelihood of A and B in the following way:

$$\begin{aligned} \Pr(A, B) &= \Pr(A|B)\Pr(B) \\ &= \Pr(B|A)\Pr(A) \end{aligned}$$

Then

$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)} \quad \text{(a 1)}$$

Thus the probability density function (pdf) of the expected return given the data equilibrium return, $\Pr(A|B)$, is given by the product of the conditional pdf of the data equilibrium return, $\Pr(B|A)$, and the prior pdf, $\Pr(A)$, which summarizes the investment manager's subjective views, in units of marginal probabilities, $\Pr(B)$, of the equilibrium returns. Therefore, Bayes Law provides a formal mechanism to synthesize subjective views with empirical realities. As new data arrive, the posterior density can play the role of a new prior, thus updating investors beliefs in this set up.

1.2 Notation

r the $n \times 1$ vector of excess returns

Σ the $n \times n$ covariance matrix

$E(\mathbf{r}) = E(\mathbf{r}_{t+1}|I_t)$ the $n \times 1$ vector of investor-expected excess returns

$\boldsymbol{\pi}$ the CAPM equilibrium excess returns, such that

$$\begin{aligned} \pi &= \boldsymbol{\beta}\, r_m \\ &= \boldsymbol{\beta}\, w_m^{'}\mathbf{r} \end{aligned}$$

where w_m is the vector of capitalization weights, and $\boldsymbol{\beta} = \frac{\text{Cov}\left(\mathbf{r}, w_m^{'}\mathbf{r}\right)}{\text{Var}\left(w_m^{'}\mathbf{r}\right)}$.

1.3 Model Assumptions

We now intend to make the appropriate assumptions to construct the composite equation (a1) which in our established notation would write

$$\Pr\left(E(\mathbf{r})|\boldsymbol{\pi}\right) = \frac{\Pr\left(\boldsymbol{\pi}|E(\mathbf{r})\right)\Pr\left(E(\mathbf{r})\right)}{\Pr\left(\boldsymbol{\pi}\right)} \qquad \text{(a 2)}$$

We shall assume that prior beliefs in $\Pr\left(E(\mathbf{r})\right)$, shall take the form of k linear constraints on the vector of n expected returns $E(\mathbf{r})$ which, can be expressed with a $k \times n$ matrix P such that

$$P\ E(\mathbf{r}) = \mathbf{q} + \mathbf{v}$$

where $\mathbf{v} \sim \mathbf{N}\left(\mathbf{0}, \boldsymbol{\Omega}\right)$ and $\boldsymbol{\Omega}$ is a $k \times k$ diagonal covariance matrix. Then

$$P\ E(\mathbf{r}) \sim \mathbf{N}\left(\mathbf{q}, \boldsymbol{\Omega}\right) \qquad \text{(a 3)}$$

The existence of an error vector $\mathbf{v}$ signifies the existence of uncertain views, However, the normality assumption coupled with a diagonal $\boldsymbol{\Omega}$ implies that the investment manager's subjective views are formed independently of each other. As the diagonal elements of $\mathbf{\Omega}$ string to zero, opinions are formed exactly (with certainty) by $P\ E(\mathbf{r}) = \mathbf{q}$. Note that $P, \mathbf{\Omega}$ and $\mathbf{q}$ are known by the investor.

The probability density function of the data equilibrium returns conditional on the investor's of prior beliefs, is assumed to be

$$\pi|\mathbf{E}(\mathbf{r}) \sim \mathbf{N}\left(\mathbf{E}(\mathbf{r}), \boldsymbol{\tau}\mathbf{\Sigma}\right) \tag{a 4}$$

The fact that $E(\boldsymbol{\pi}) = \mathbf{E}(\mathbf{r})$ reflects the assumption of homogeneous views of all the investors in a CAPM-type world. Also, the scalar τ is a known quantity to the investor that scales the historical covariance matrix $\mathbf{\Sigma}$.

Finally, the marginal density function of data equilibrium returns, $\Pr\left(\boldsymbol{\pi}\right)$, is a constant that will be absorbed into the integrating constant of the $\Pr\left(E(\mathbf{r})|\boldsymbol{\pi}\right)$.

2 Certain Prior Beliefs on Expected Returns

In this case the certainty regarding prior beliefs corresponds to zero standard deviation. Thus, the investors views are expressed as an exact relationship which will simply form a constraint in an optimization problem. In particular, we have

$$\min_{E(\mathbf{r})} \left(E(\mathbf{r}) - \pi\right)' \tau\Sigma\ \left(E(\mathbf{r}) - \pi\right)$$

$$\text{s.t. } PE(\mathbf{r}) = \mathbf{q}$$

Proposition 1 *The optimal predictor of $E(\mathbf{r})$ that minimizes its variance around equilibrium returns π and satisfies k exact linear belief constraints is* *given by*

$$\widehat{E(\mathbf{r})} = \pi + \Sigma^{-1}P'\left(P\Sigma^{-1}P'\right)^{-1}(\mathbf{q}{-}P\pi)$$

Proof:

This is a conventional linearly constrained least-squares problem and thus admits a closed-form solution for $E(\mathbf{r})$. In particular,we can form a Lagrangian function

$$\begin{aligned} L &= (E(\mathbf{r}) - \pi)' \tau\Sigma\ (E(\mathbf{r}) - \pi) - \lambda (PE(\mathbf{r}) - \mathbf{q}) \\ &= E(\mathbf{r})'\tau\Sigma\ E(\mathbf{r}) - E(\mathbf{r})'\tau\Sigma\ \pi - \pi'\tau\Sigma\ E(\mathbf{r}) + \pi'\tau\Sigma\ \pi \\ &\quad -\lambda\ PE(\mathbf{r}) + \lambda\ \mathbf{q} \end{aligned}$$

The f.o.c.'s will be

$$\begin{aligned} \frac{\partial L}{\partial E(\mathbf{r})} &= \tau\Sigma' E(\mathbf{r}) + \tau\Sigma E(\mathbf{r}) - \tau\Sigma\ \pi - \tau\Sigma'\pi = 0 \\ &= 2\tau\Sigma E(\mathbf{r}) - 2\tau\Sigma\ \pi - P'\lambda = 0 \qquad (1) \end{aligned}$$

$$\frac{\partial L}{\partial \lambda} = PE(\mathbf{r}) - \mathbf{q} = \mathbf{0} \qquad (2)$$

Solving equation (1) wrt $E(\mathbf{r})$ we obtain

$$E(\mathbf{r}) = \pi + \frac{1}{2\tau}\lambda\Sigma^{-1}P'$$

then substitute to equation (2) to obtain the value of the Lagrange multiplier

$$\lambda = \left(P\Sigma^{-1}P'\right)^{-1} 2\tau\ \ (q - P\pi)$$

Substituting λ back into (1) we obtain the optimal value for $E(\mathbf{r})$

$$E(\mathbf{r}) = \pi + \Sigma^{-1}P'\left(P\Sigma^{-1}P'\right)^{-1}(q - P\pi)$$

□

Corollary 2 *Non-existence of prior beliefs implies that $E(\mathbf{r})$ collapses to the data equilibrium return π.*

Proof: Set $P = \mathbf{0}$ in $E(\mathbf{r}) = \pi + \Sigma^{-1}P' \left(P\Sigma^{-1}P'\right)^{-1}(q - P\pi)$.□

3 Uncertain Prior Beliefs on Expected Returns

When the investor forms prior beliefs with a degree of uncertainty, this is signified in the non-zero value of the diagonal elements of the Ω matrix. Using assumptions (a3) and (a4) in equation (a2) we obtain the following result:

Proposition 3 *The posterior probability density function pdf($E(\mathbf{r})|\boldsymbol{\pi}$) is multivariate normal with mean*

$$\left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\boldsymbol{\pi} + P'\Omega^{-1}\mathbf{q}\right]$$

and variance

$$\left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}$$

Proof:

Assumptions (a3) and (a4) state respectively that

$$\text{pdf}\left(PE(\mathbf{r})\right) = \frac{k}{\sqrt{2\pi_c\left|\Omega\right|}}\exp\left(-\frac{1}{2}\left(PE(\mathbf{r}) - \mathbf{q}\right)'\Omega^{-1}\left(PE(\mathbf{r}) - \mathbf{q}\right)\right)$$

and

$$\text{pdf}\left(\pi|E(\mathbf{r})\right) = \frac{k}{\sqrt{2\pi_c\left|\tau\Sigma\right|}}\exp\left(-\frac{1}{2}\left(\pi - E(\mathbf{r})\right)'(\tau\Sigma)^{-1}\left(\pi - E(\mathbf{r})\right)\right)$$

From (a2) we know that

$$\Pr\left(E(\mathbf{r})|\boldsymbol{\pi}\right) = \frac{\Pr\left(\boldsymbol{\pi}|E(\mathbf{r})\right)\Pr\left(E(\mathbf{r})\right)}{\Pr\left(\boldsymbol{\pi}\right)}$$

Substituting the pdf's, the posterior density will be proportional to

$$\exp\left(-\frac{1}{2}\left(\pi - E(\mathbf{r})\right)'\left(\tau\Sigma\right)^{-1}\left(\pi - E(\mathbf{r})\right) - \frac{1}{2}\left(PE(\mathbf{r}) - \mathbf{q}\right)'\Omega^{-1}\left(PE(\mathbf{r}) - \mathbf{q}\right)\right)$$

which can be written as

$$\begin{aligned} & \exp\left(-\frac{1}{2}\left[E(\mathbf{r})'H\ E(\mathbf{r}) - 2C'E(\mathbf{r}) + A\right]\right) \\ = & \exp\left(-\frac{1}{2}\left[E(\mathbf{r})'H'H\ H^{-1}E(\mathbf{r}) - 2C'H^{-1}H\ E(\mathbf{r}) + A\right]\right) \\ = & \exp\left(-\frac{1}{2}\left[\left(H\ E(\mathbf{r}) - C\right)'H^{-1}\left(H\ E(\mathbf{r}) - C\right) - C'\ H^{-1}C + A\right]\right) \\ = & \exp\left(-\frac{1}{2}\left[A - C'\ H^{-1}C\right]\right) \times \exp\left(-\frac{1}{2}\left(H\ E(\mathbf{r}) - C\right)'H^{-1}\left(H\ E(\mathbf{r}) - C\right)\right) \end{aligned}$$

where

$$\begin{aligned} H &= \left(\tau\Sigma\right)^{-1} + P'\Omega^{-1}P \\ C &= \left(\tau\Sigma\right)^{-1}\boldsymbol{\pi} + P'\Omega^{-1}\mathbf{q} \\ A &= \boldsymbol{\pi}'\left(\tau\Sigma\right)^{-1}\boldsymbol{\pi} + \mathbf{q}'\Omega^{-1}\mathbf{q} \end{aligned}$$

Thus, the term $\exp\left(-\frac{1}{2}\left[A - C'\ H^{-1}C\right]\right)$ and the denominator pdf($\boldsymbol{\pi}$) which is not modelled will be absorbed into the integrating constant for the posterior pdf. Hence the result follows immediately.□

Corollary 4 *As uncertainty about in investor views reduces (investor becomes more confident about his/her views), the expected return approaches the deterministic case, in which*

$$E(\mathbf{r}) = \pi + \Sigma^{-1}P'\left(P\Sigma^{-1}P'\right)^{-1}\left(q - P\pi\right)$$

4 Interpretation of Proposition 3

We have proved that $E(\mathbf{r})|\boldsymbol{\pi}$ has posterior mean

$$\left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\boldsymbol{\pi} + P'\Omega^{-1}\mathbf{q}\right]$$

which can be written as

$$\left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\boldsymbol{\pi} + \left(P'\Omega^{-1}P\right)\underbrace{\left(P'P\right)^{-1}P'\mathbf{q}}_{n\times 1}\right]$$

Also we know that

$$P\ E(\mathbf{r}) = \mathbf{q} + \mathbf{v}$$

or that

$$\mathbf{q} = P\ E(\mathbf{r}) - \mathbf{v}$$

which can be seen as a "regression" of $\mathbf{q}$ on P, with $E(\mathbf{r})$ being the vector of unknown (to be estimated) coefficients[1]. Then

$$\left(P'P\right)^{-1}P'\mathbf{q}$$

can be interpreted as the least squares estimate of expected returns, $E(\mathbf{r})$, according to investors views

$$\left(P'P\right)^{-1}P'\mathbf{q} = \widehat{E(\mathbf{r})}$$

Thus, Proposition 3 can be restated in the form

$$\text{mean of } E(\mathbf{r})|\boldsymbol{\pi} = \left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\boldsymbol{\pi} + \left(P'\Omega^{-1}P\right)\widehat{E(\mathbf{r})}\right]$$

[1] of course we need $k > n$

which makes clear how subjective views are combined with data-equilibrium. The term in the second square brackets is a weighted average of data equilibrium $\boldsymbol{\pi}$ and the least squares estimate of expected returns, $\widehat{E(\mathbf{r})}$, according to investors views, the (vector) weights being $(\tau\Sigma)^{-1}$ and $(P'\Omega^{-1}P)$ respectively.

If the distribution of expected returns around the data equilibrium $\boldsymbol{\pi}$ is tight, i.e. $\tau\Sigma$ small, then $(\tau\Sigma)^{-1}$ will be large and more weight will be put to $\boldsymbol{\pi}$. If the investor is confident about his/views then Ω is small, resulting in a large $P'\Omega^{-1}P$ which puts more weight on the least squares views $\widehat{E(\mathbf{r})}$.

5 Optimal Asset Allocation Recommendations

As we have established in the mean-variance analysis, the vector of weights of any frontier portfolio can be written as a mixture of any two distinct frontier portfolios. In fact, these can be the globally-minimum-variance-portfolio, $\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}$, and the original portfolio's zero-covariance portfolio. But since the latter is a frontier portfolio, it should be a function of the vector of expected returns. The Black-Litterman model provides a framework to update subjective views on expected returns with data equilibrium returns and makes possible to tactically allocate funds according to this process.

References

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